VIBRATION ANALYSIS OF ISOTROPIC AND POLAR ORTHOTROPIC CIRCULAR AND ANNULAR PLATES USING FINITE ELEMENTS

by

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DEDICATED

TO

THE LOVING MEMORY OF

MY PARENTS

(Late Sri C. Joseph and Late Smt. P. V. Thresia)

ME - 1984 - M - BEN - VIB

CERTIFICATE

This is to certify that the thesis entitled, 'VIBRATION ANALYSIS OF ISOTROPIC AND POLAR ORTHOTROPIC CIRCULAR AND ANNULAR PLATES USING FINITE ELEMENTS', by Mr. J. Benjamin has been carried out under my supervision and has not been submitted elsewhere for a degree.

July, 1984

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NOMENCLATURE

a	Outer radius
a_i, $i=1,2,..$	Polynomial coefficients
b	Inner radius
$[B]$	Matrix relating polynomial coefficients a_i and and the element nodal deflections
$\{c\}$, $\lfloor c \rfloor$	Column and row matrices of polynomial coefficients respectively
C	Constant
$\{d\}$, $\lfloor d \rfloor$	Column and row matrices defined in eqn. (2.2), respectively
D	Flexural rigidity of isotropic plate
$D_r, D_\theta, D_{r\theta}$	Flexural rigidities of polar orthotropic plate in radial, tangential and r-θ directions, respectively
$[E]$	Matrix relating the element nodal deflections and polynomial coefficients
$[F]$	Matrix defined in eqn. (2.9)
h	Thickness of the plate
$[k]$, $[K]$	Stiffness matrices
l	Length of the element in radial direction
$[m]$, $[M]$	Mass matrices
m	Number of nodal diameters
n	Number of nodal circles
NE	Number of elements
r, θ, z	Polar coordinates for plate
$\{s\}$, $\lfloor s \rfloor$	Column and row matrices defined in eqn.(2.5), respectively

t	Time coordinate
T	Potential energy in an element
U, U_1	Kinetic energy in an element of isotropic and polar orthotropic plates, respectively
$[V], [V_1]$	Matrices defined by eqns. (2.4) and (3.4), respectively
w	Axial deflection
$\bar{w}$	Axial deflection along the reference antinode, Fig. 2.1
$\bar{w}_1, \bar{w}_2$	Axial deflections at nodes 1 and 2 respectively, Fig. 2.1
$\dot{w}$	Differentiation of w with respect to time
ϕ	Radial rotation
$\bar{\phi}$	Radial rotation along the reference antinode Fig. 2.1
$\bar{\phi}_1, \bar{\phi}_2$	Radial rotations at nodes 1 and 2, respectively, Fig. 2.1
ν, ν_θ	Poisson's ratio for isotropic and polar orthotropic plates, respectively
	Plate material density
ω	Natural frequency
S, F, C	Combination of any two denotes edge conditions at outer and inner edges, respectively
i, j i=0,1,2,3 j=0,1,2,3	Denote number of nodal circles and nodal diameters, respectively

ABSTRACT

The finite element method is applied to the vibration analysis of isotropic and polar orthotropic plates. Annular finite elements having four degrees of freedom, these being the displacements and slopes at inner and outer radii, are used to obtain the natural frequencies of all the nine combinations of clamped, free and simply supported edge conditions of annular plates. Using the annular elements itself, the natural frequencies are obtained for solid circular plates having clamped, free or simply supported boundaries.

CHAPTER 1

INTRODUCTION

1.1 GENERAL

Vibration characteristics of circular plates are of interest in design since circular plates with and without holes are commonly used structural elements. Fibre-reinforced composite materials are attractive for these structural elements because of their high stiffness, relatively low density and highly controllable anisotropic properties. Moreover polar orthotropic composites have great potential for the development of flywheels for various applications. The transverse flexural vibration can constitute a major problem in designing flywheels. Analysis of these polar orthotropic materials generally defies the possibility of exact solution and necessitates the use of approximate methods. Exact solutions are possible only for highly restricted classes of problems; the restrictions tend to be more severe for vibration problems than for static problems.

In the last 20 years high speed electronic digital computers and finite element methods suitable for computer aided analysis have been rapidly developed. The governing differential equations describing the behaviour of any continuum are derived considering infinitesimal elements of the continuum.

A finite number of these infinitesimal elements build up the continuum which thus has infinite degrees of freedom. In finite element idealisation, elements of finite size are considered in place of infinitesimal elements to build up the actual continuum, and the behaviour of the continuum is sought to be represented through the behaviour of individual elements. In this idealisation each element is connected to its neighbouring elements, not continuously, but through a finite number of strategically located points or nodes on its boundary, thereby reducing the infinite degreesof freedom of the continuum to a finite number of degrees of freedom. Using the same principles that also hold good for the actual continuum, the behaviour of each element is exactly or approximately determined in terms of the variables at the connecting nodes. If the required continuity of the variables is maintained between the elements, then the finite element solution is expected to converge towards the true solution with greater number of smaller elements. So, in order to study the dynamic response characteristics of polar orthotropic plates the finite element method is found to be less tedious compared to other approximate analyses and hence in this present work finite element method is used for the analysis.

1.2 REVIEW OF PREVIOUS WORK

In the literature, one finds an extensive treatment of the problem of fluxural vibration of isotropic circular plates and,

to a lesser extent, annular plate as well. The problem of vibration of isotropic circular plates having free edges was investigated long back by Kirchhoff, Lamb and Rayleigh as cited [1], using Poisson-Kirchhoff theory. Timoshenko [2] used the energy method for solving the case of isotropic plate with clamped edges. Thein Wah [1] investigated the case of simply supported edge taking Poisson-Kirchhoff theory as the basic equation. Numerical results are also given for frequency parameter for modes of vibration consisting of nodal circles and nodal diameters. An extensive study of uniform annular isotropic plates has been done by Vogel et al [3]. Here the classical theory of fluxural motions of elastic plates was used to determine the natural frequencies for all nine possible combinations of clamped, free and simply supported edge conditions and for different hole sizes. An investigation was also made into the effects of Poisson's ratio on frequency parameter by taking different values of Poisson's ratio. No significant variation was noted in the frequency parameter for different values of Poisson's ratio. Hence, in reference [3], the analysis has been done by taking Poisson's ratio equal to 0.3.

In the text by Zienkiewicz [4] several applications of the finite element method (FEM) to the vibration analysis of circular plates are given. The simplest element shapes for plate problems are obviously a triangle and rectangle with three and four

nodes, respectively [5]. Rectangular elements are somewhat limited in their boundary shape applicability, whereas triangular elements may be used to represent any boundary. However, in our present problem, the use of triangular elements means that the curved boundary is being approximated by a series of straight line segments. In order to overcome this approximation Ergatoudis [6] has developed quadrilateral elements with various curved sides. Here , the basic shape of the element chosen is quadrilateral, the sides of which can, however, be distorted in a prescribed way. M.D. Olson et al [7] have developed two plate bending finite elements in polar coordinates mainly to study the vibration problem of circular plates. The first element has three nodal corners and is in the shape of a sector of a circle. The second element has four nodal corners and is in the shape of a segment of an annulus. The transverse displacement and two slopes are used as the degrees of freedom at each element nodal corner, resulting in nine degrees of freedom for circular sector element and twelve degrees of freedom for annular sector element. Non-dimensional frequency parameter for complete circular plate with clamped, free and simply supported boundaries are tabulated and compared with exact values. In a recent paper by Kirkhope et al [8] annular finite elements are derived which describe the bending and stretching of thin rotating discs. These elements incorporate the desired number of diametral nodes in their dynamic

deflection function, and allow for any specified thickness variation in the radial direction.

Even though the literature contain a good number of treatment for the vibration of isotropic plates, the vibration analysis of orthotropic circular plate have, however, apparently received scant mention in the literature. Kirmser et al. [9], Pandalai et al. [10],Joung [11], Minkarah et al [12], Bellini et al. [13], Huang [14] and Nowinski et al.[15] have attempted some of the problems of polar orthotropic bodies subjected to dynamic loading. It is to be noted that the existence of material singularity at the centre has been reported by many of them. This means that the circumferential and radial moduli at the centre cannot be different for polar orthotropic materials. In order to include the aforesaid material singularity at the centre some investigators have assumed an isotropic core of uniform property while dealing with the problem of circular disc.

Natural frequencies corresponding to axisymmetric mode of vibration of polar orthotropic annular plates have been obtained by Vijayakumar et al. [16] for various combinations of clamped, free and simply supported edge conditions. Classical Rayleigh-Ritz method with simple polynomials as admissible functions has been used to get the least eigenvalues. In an attempt to determine the natural frequencies of higher modes of vibration, Ramaiah et al. [17] have used the classical

Reyleigh-Ritz method by introducing a coordinate transformation. Thus they have obtained frequencies corresponding to the axisymmetric modes and modes with one and two nodal diameters. However, they themselves have admitted that the application of this method is generally cumbersome since the method involves solutions of many simultaneous equations, if reliable estimates of the frequencies are desired. It may be due to this difficulty that the numerical results are given only for simply supported-free, clamped-free and free-free annular plates and this also for two selected ratio of radii 0.1 and 0.5. They have determined these frequencies for two different cases of rigidity ratio 0.5 and 2.0. In another recent investigation by Ramaiah et al. [18] the frequencies corresponding to higher modes of vibrating polar orthotropic annular plates have been obtained based on an assumption that the radial bending moment is zero at a nodal circle. No numerical results are given.

1.3 PRESENT WORK

What is lacking in the literature, is a thorough numerical evaluation of natural frequencies of polar orthotropic circular plates with simply supported, clamped and free edge conditions, and frequencies for polar orthotropic annular plates with all the nine combinations of simply supported, clamped and free edge conditions. To evaluate these natural frequencies, in this present work finite element method is

used which is found to be more efficient when compared with all other methods.

The study of natural frequencies of polar orthotropic circular as well as annular plates is the concern of the present work. The FEM is applied to the vibration analysis and the natural frequencies are computed by means of a finite element program. Kirkhope et al. [8] have recently developed annular finite elements to describe the bending and stretching of thin rotating isotropic discs. These elements allow for any specified thickness variation in the radial direction. In the present work this approach has been adopted to determine natural frequencies of non-rotating uniform isotropic plates and its applicability is extended to polar orthotropic plates also. In order to formulate the eigenvalue problems associated with the vibration analyses, the element stiffness and mass matrices are derived for isotropic as well as orthotropic plates by appropriate manipulation of the chosen displacement function.

In Chapter 2 the derivation of stiffness matrix for isotropic plate is given. It is derived by putting the assumed displacement function in the expression for strain energy. The mass matrix is derived by substituting the assumed displacement function in the equation for ~~potential~~ Kinetic energy of the plate. Non-dimensional frequency parameters are obtained for all the nine possible combinations of clamped, free and simply supported

edge conditions of the annular plates. The results are compared with exact values and graphs are plotted to find the variation of frequency with hole size. The frequencies of solid circular plates with clamped, free and simply supported edges are also calculated and compared with exact values.

In Chapter 3, the stiffness matrix for polar orthotropic annular plate is derived. The expression for mass matrix of polar orthotropic plate will be the same as that of isotropic plate because the mass matrix is independent of material properties D_r, D_θ and $D_{r\theta}$. The frequencies of polar orthotropic circular plates for all the three edge conditions are determined. For polar orthotropic annular plates the frequencies are obtained for all the nine cases of edge conditions and for different hole sizes. These frequencies are calculated for a wide range of rigidity ratio and are compared with the available values.

Results and discussion are given in Chapter 4 and in Chapter 5, some general conclusions are summarized and possible extensions of the present work are discussed.

CHAPTER 2

GENERAL FORMULATION OF VIBRATION PROBLEM OF CIRCULAR AND ANNULAR ISOTROPIC PLATES

2.1 INTRODUCTION

One of the problems in structural dynamics is the vibration of circular and annular plates. Exact solutions for circular[1] and annular plates [3] for various edge conditions are readily available. One of the several approximate methods used to study the vibration problem is finite element method (FEM). Olson et al [7] used circular sector elements having nine degrees of freedom and annular sector elements having twelve degrees of freedom to estimate the natural frequencies of isotropic plates. These elements result in a very large number of degrees of freedom if good accuracy is desired. The annular element derived by Kirkhope et al [8] for rotating disc of nonuniform profile has only four degrees of freedom. It has been observed that even with two annular elements, the results obtained are very close to the exact values. Hence, in the present chapter, natural frequencies of isotropic circular and annular plates are determined using annular finite elements having four degrees of freedom.

2.2 FORMULATION OF THE ANNULAR ELEMENT

The vibration of circular plates is characterised by modes having integer numbers of nodal diameters and nodal circles. For each nodal diameter configuration there exists a family of circular nodes with corresponding natural frequencies. Fig. 2.1 shows the annular finite element with associated degrees of freedom and diametral nodes. The finite element has two nodes, with 2 degrees of freedom/node. They are : the axial deflection ($\bar{w}$), and radial rotation ($\bar{\phi}$). The rotation ϕ is defined as $\phi = \frac{\partial w}{\partial r}$ and the bars over w and ϕ indicate that these are taken along the reference antinode. Once, $\bar{w}$, and $\bar{\phi}$, at the reference antinode, where θ is taken as zero, are specified, the deflection and slope at any other point at an angle θ from the reference antinode, as shown in Fig. 2.1, can be expressed as $\bar{w}$ cos$m\theta$ and $\bar{\phi}$ cos$m\theta$. Hence, a suitable deflection function for $\bar{w}$ of the plate along the radial direction only remains to be chosen.

The exact solution to the problem is of the form $w(r,\theta) = F(r) \cos m\theta$. So we can choose a deflection function of the element as $w(r,\theta) = F(r) \cos m\theta$, where $F(r)$ is the deflection of the vibrating plate along the reference antinode and can be approximated by a polynomial of r. Use of $\cos m\theta$ in the expression incorporates the desired number of nodal diameters m in the element.

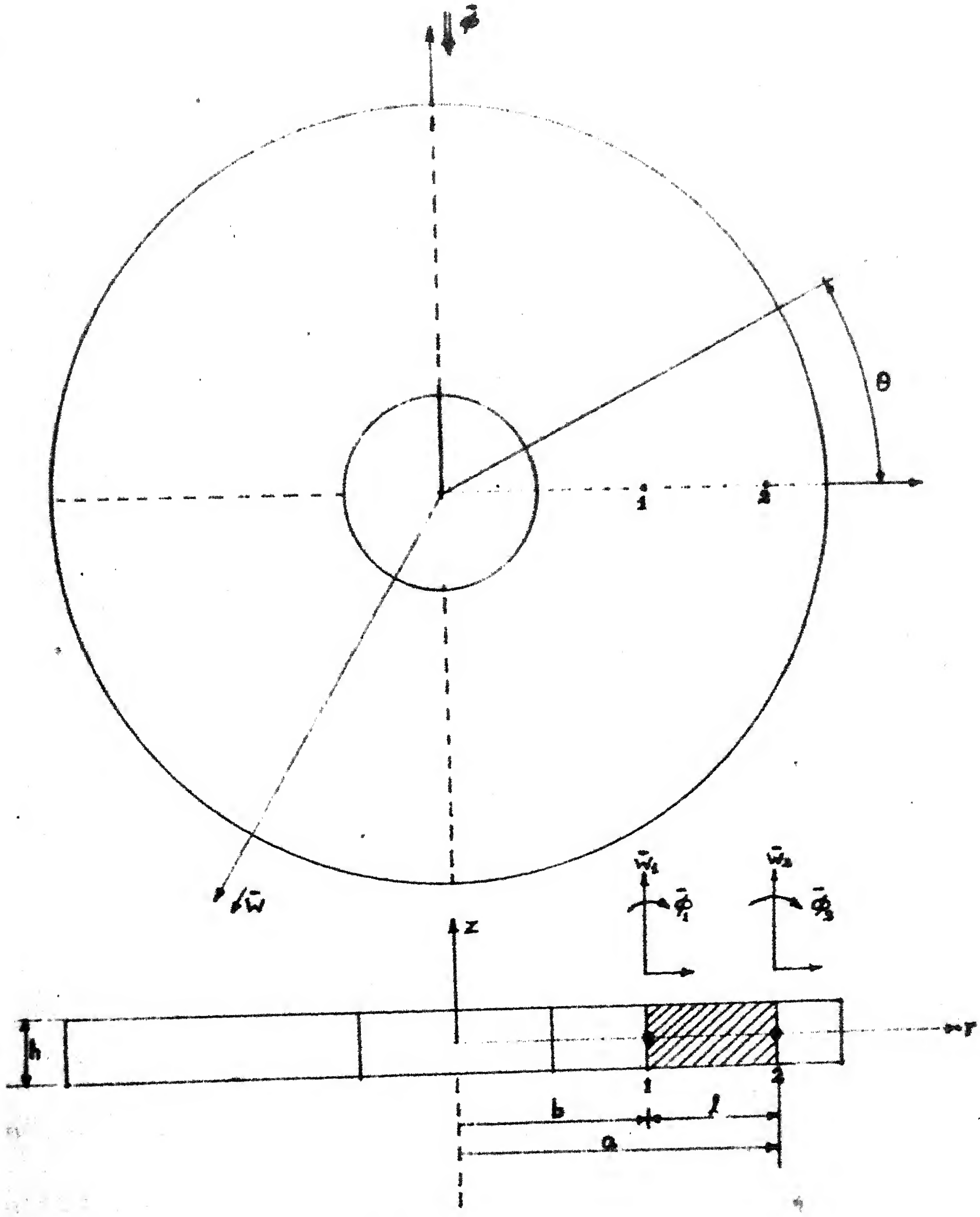

Fig. 2.1 Annular element with m = 2

2.2.1 Stiffness Matrix for Isotropic Plates

For classical bending of isotropic plates, the strain energy, for an element in polar co-ordinates is given by [2],

$$U = \frac{D}{2} \int_0^{2\pi} \int_b^a \left[\left(\frac{\partial^2 w}{\partial r^2}\right)^2 + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 + 2\nu \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + 2(1-\nu)\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)^2 \right] r\,dr\,d\theta \qquad (2.1)$$

which can be written as

$$U = \frac{D}{2} \int_0^{2\pi} \int_b^a \lfloor d \rfloor [V] \{d\}\; r\,dr\,d\theta \qquad (2.2)$$

where,

$$\lfloor d \rfloor = \left\lfloor \left(\frac{\partial^2 w}{\partial r^2}\right) \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) \left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) \right\rfloor \qquad (2.3)$$

and

$$[V] = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 2(1-\nu) \end{bmatrix} \qquad (2.4)$$

where is Poisson's ratio.

For the annular element the assumed deflection function can be written as,

$w(r,\theta) = F(r) \cos m\theta$, where $F(r)$ is a polynomial of r,

$$w(r,\theta) = (a_1 + a_2 r + a_3 r^2 + a_4 r^3) \cos m\theta$$

$$w(r,\theta) = \lfloor s \rfloor \{c\} \cos m\theta \qquad (2.5)$$

where,

$$\lfloor s \rfloor = \lfloor 1 \quad r \quad r^2 \quad r^3 \rfloor$$

and

$$\lfloor c \rfloor = \lfloor a_1 \quad a_2 \quad a_3 \quad a_4 \rfloor$$

Since $\theta = 0$ along the reference antinode, the lateral deflection and radial slope along this antinode are,

$$\bar{w}(r) = a_1 + a_2 r + a_3 r^2 + a_4 r^3$$

$$\bar{\phi}(r) = a_2 + 2a_3 r + 3a_4 r^2$$

Referring to Fig. 2.1

$$\bar{w}_1 = a_1 + a_2 b + a_3 b^2 + a_4 b^3$$

$$\bar{\phi}_1 = a_2 + 2a_3 b + 3a_4 b^2$$

$$\bar{w}_2 = a_1 + a_2 a + a_3 a^2 + a_4 a^3$$

$$\bar{\phi}_2 = a_2 + 2a_3 a + 3a_4 a^2$$

Let $\{\bar{w}\}$ be

$$\begin{Bmatrix} \bar{w}_1 \\ \bar{\phi}_1 \\ \bar{w}_2 \\ \bar{\phi}_2 \end{Bmatrix}$$

then $\{\bar{w}\} = [E]\,\{c\}$ (2.6)

where

$$[E] = \begin{bmatrix} 1 & b & b^2 & b^3 \\ 0 & 1 & 2b & 3b^2 \\ 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \end{bmatrix}$$

From eqn. (2.6)

$$\{c\} = [E]^{-1} \{\bar{w}\}$$

$$\{c\} = [B] \{\bar{w}\} \tag{2.7}$$

where $[B] = [E]^{-1}$

Putting this value of $\{c\}$ in eqn. (2.5)

$$w(r,\theta) = \lfloor s \rfloor [B] \{\bar{w}\} \cos m\theta \tag{2.8}$$

$$[B] =$$

$$\begin{bmatrix} \frac{a^2(a-3b)}{l^3} & \frac{a^2 b}{l^2} & \frac{b^2(3a-b)}{l^3} & \frac{ab^2}{l^2} \\ \frac{6ab}{l^3} & -\frac{a(a+2b)}{l^2} & -\frac{6ab}{l^3} & -\frac{b(2a+b)}{l^2} \\ -\frac{3(a+b)}{l^3} & \frac{(2a+b)}{l^2} & \frac{3(a+b)}{l^3} & \frac{(a+2b)}{l^2} \\ \frac{2}{l^3} & -\frac{1}{l^2} & -\frac{2}{l^3} & -\frac{1}{l^2} \end{bmatrix}$$

Putting this value of $w(r,\theta)$ in equation (2,3)

$$\{d\} = \begin{bmatrix} 0 & 0 & 2\alpha & 6r\alpha \\ -\frac{m^2\alpha}{r^2} & -\frac{\alpha}{r}(m^2-1) & -\alpha(m^2-2) & -\alpha r(m^2-3) \\ \frac{m\beta}{r^2} & 0 & -m\beta & -2m\beta r \end{bmatrix} [B]\{\overline{w}\}$$

where $\alpha = \cos m\theta$

$\beta = \sin m\theta$

$$\{d\} = [F]\,[B]\{\overline{w}\} \qquad (2.9)$$

Substituting this value of $\{d\}$ in eqn. (2.2)

$$U = \frac{D}{2}\int_o^{2\pi}\int_b^a \lfloor\overline{w}\rfloor\,[B]^T[F]^T\,[V]\,[F]\,[B]\{\overline{w}\}\, r dr d\theta$$

$$= \frac{D}{2}\lfloor\overline{w}\rfloor\,[B]^T\,[\int_o^{2\pi}\int_b^a [F]^T\,[V]\,[F]\, r dr d\theta]\,[B]\,\{\overline{w}\}$$

Stiffness matrix is given by

$$[K] = [B]^T\,[\int_o^{2\pi}\int_b^a D[F]^T\,[V]\,[F]\, r dr d\theta]\,[B]$$

$$[K] = [B]^T\,[k]\,[B]$$

where $[k] = D\int_o^{2\pi}\int_b^a [F]^T\,[V]\,[F]\, r dr d\theta$.

$$[k] = D\int_0^{2\pi}\int_b^a \begin{bmatrix} 0 & \frac{m^2\alpha}{r^2} & \frac{m\beta}{r^2} \\ 0 & \frac{\alpha(m^2-1)}{r} & 0 \\ -2\alpha & \alpha(m^2-2) & -m\beta \\ -6\alpha r & \alpha r(m^2-3) & -2m\beta r \end{bmatrix}$$

$$\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & -2\alpha & -6\alpha r \\ \frac{m^2\alpha}{r^2} & \frac{\alpha}{r}(m^2-1) & \alpha(m^2-2) & \alpha r(m^2-3) \\ \frac{m\beta}{r^2} & 0 & -m\beta & -2m\beta r \end{bmatrix}$$

$$x(r dr d\theta)\ .$$

$$= D\int_0^{2\pi}\int_b^a \begin{bmatrix} \frac{\nu\alpha m^2}{r^2} & \frac{m^2\alpha}{r^2} & 2(1-\ )\frac{m\beta}{r^2} \\ \frac{\nu\alpha(m^2-1)}{r} & \frac{\alpha}{r}(m^2-1) & 0 \\ -2\alpha+\nu\alpha(m^2-2) & -2\nu\alpha+\alpha(m^2-2) & -2m\beta(1-\nu) \\ -6\alpha r+\nu\alpha r(m^2-3) & -6\nu\alpha r+\alpha r(m^2-3) & -4m\beta r(1-\nu) \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -2\alpha & -6\alpha r \\ \frac{m^2\alpha}{r^2} & \frac{\alpha}{r}(m^2-1) & \alpha(m^2-2) & \alpha r(m^2-3) \\ \frac{m\beta}{r^2} & 0 & -m\beta & -2m\beta r \end{bmatrix} r dr d\theta$$

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

$$k_{11} = D\int_0^{2\pi}\int_b^a \left(\frac{m^4\alpha^2}{r^4} + 2(1-\nu)\frac{m^2\beta^2}{r^4}\right) r\,dr\,d\theta \Rightarrow 2\pi \quad \text{if} \quad m = o$$

$$= D\int_0^{2\pi} (m^4\alpha^2 + 2(1-\nu)m^2\beta^2)\left[\frac{r^{-2}}{-2}\right]_b^a d\theta \qquad \text{for } mm \geq 1$$

$$= D\frac{1}{2}\frac{(a^2-b^2)}{(ab)^2} \quad \pi(m^4+2(1-\nu)m^2)$$

$$\boxed{k_{11} = C\pi D(m^4+2m^2-2\nu m^2)\frac{(a^2-b^2)}{2(ab)^2}}$$

$C = 2$ if $m = 0$

$C = 1$ if $m \geq 1$

$$k_{12} = D\int_0^{2\pi}\int_b^a \frac{\alpha^2}{r^3} m^2(m^2-1) \; r\,dr\,d\theta$$

$$\boxed{k_{12} = DC\pi(m^4-m^2)\frac{(a-b)}{ab}}$$

$$k_{13} = D\int_0^{2\pi}\int_b^a \left(-2\nu\frac{\alpha^2 m^2}{r^2} + \frac{\alpha^2(m^4-2m^2)}{r^2} - \frac{2(1-\quad)m^2\beta^2}{r^2}\right) r\,dr\,d\theta$$

$$= C\pi D(-2\nu m^2+(m^4-2m^2)-2(1-\nu)m^2) \quad \ln(r_2/r_1)$$

$$\boxed{k_{13} = D\,C\pi(m^4-4m^2) \quad \ln(a/b)}$$

$$k_{14} = D\int_0^{2\pi}\int_b^a \left(-6\frac{\gamma\alpha^2 m^2}{r} + \frac{m^2\alpha^2}{r}(m^2-3) - \frac{4m^2\beta^2(1-\gamma)}{r}\right) rdrd\theta$$

$$= DC\pi(-6\gamma m^2+(m^4-3m^2)-4m^2+4\gamma m^2)(a-b)$$

$$\boxed{k_{14} = C\pi D(m^4-7m^2-2\gamma m^2)(a-b)}$$

$$k_{21} = D\int_0^{2\pi}\int_b^a \frac{\alpha^2(m^4-m^2)}{r^3}\, rdrd\theta \,.$$

$$= DC\pi(m^4-m^2)\left(\frac{a-b}{ab}\right)$$

$$\boxed{k_{21} = k_{12}}$$

$$k_{22} = D\int_0^{2\pi}\int_b^a (m^2-1)^2\, rdrd\theta$$

$$\boxed{k_{22} = DC\pi(m^4-2m^2+1)\quad \ln(a/b)}$$

$$k_{23} = D\int_0^{2\pi}\int_b^a \left(-\frac{2\alpha^2}{r}\gamma\,(m^2-1)+\frac{\alpha^2}{r}(m^2-2)(m^2-1)\right) rdrd\theta$$

$$= C\pi D(-2\gamma(m^2-1) + (m^2-2)(m^2-1))(a-b)$$

$$\boxed{k_{23} = C\pi D(m^4-3m^2-2\gamma m^2+2\gamma+2)(a-b)}$$

$$k_{24} = D\int_0^{2\pi}\int_b^a (-6\alpha^2\gamma(m^2-1)+\alpha^2(m^2-1)(m^2-3))\, rdrd\theta$$

$$= C\pi D(-6\gamma(m^2-1)+(m^2-1)(m^2-3))\left(\frac{a^2-b^2}{2}\right)$$

$$\boxed{k_{24} = DC\pi(m^4-4m^2-6\gamma m^2+6\gamma+3)\left(\frac{a^2-b^2}{2}\right)}$$

$$k_{31} = \mathcal{D}\int_0^{2\pi}\int_b^a \left(-2\nu\frac{\alpha^2 m^2}{r^2} + \frac{\alpha^2}{r^2}(m^4-2m^2) - \frac{2m^2\beta^2}{r^2}(1-\nu)\right) r\,dr\,d\theta$$

$$= \mathcal{D}C\pi(m^4-4m^2)\ \ln(a/b)$$

$$\boxed{k_{31} = k_{13}}$$

$$k_{32} = \mathcal{D}\int_0^{2\pi}\int_b^a \left(-2\ \frac{\alpha^2}{r}(m^2-1) + \frac{\alpha^2}{r}(m^2-2)(m^2-1)\right) r\,dr\,d\theta$$

$$= \mathcal{D}C\pi(m^4-3m^2-2\quad m^2+2\quad +2)(a-b)$$

$$= k_{23}$$

$$\boxed{k_{32} = k_{23}}$$

$$k_{33} = \mathcal{D}\int_0^{2\pi}\int_b^a \left(4\alpha^2 - 2\nu\alpha^2(m^2-2) - 2\nu\alpha^2(m^2-2) + \alpha^2(m^2-2)^2\right.$$

$$\left. + 2m^2\beta^2(1-\nu)\right)\ r\,dr\,d\theta$$

$$= \mathcal{D}C\pi\left(4-2\nu(m^2-2)-2\nu(m^2-2)+(m^2-2)^2+(1-\nu)2m^2\right)\left(\frac{a^2-b^2}{2}\right)$$

$$\boxed{k_{33} = \mathcal{D}C\pi(m^4-6\nu m^2-2m^2+8\nu+8)\ \left(\frac{a^2-b^2}{2}\right)}$$

$$k_{34} = \mathcal{D}\int_0^{2\pi}\int_b^a \left(12\alpha^2 r - 6\nu\alpha^2 r(m^2-2) - 2\alpha^2\nu r(m^2-3) + \alpha^2 r(m^2-3)(m^2-2)\right.$$

$$\left. + 4(1-\nu)m^2\beta^2\right)\ r\,dr\,d\theta$$

$$= \mathcal{D}C\pi\left(12-6\nu(m^2-2)-2\nu(m^2-3)+(m^2-3)(m^2-2)+4(1-\nu)m^2\right)\left(\frac{a^3-b^3}{3}\right)$$

$$\boxed{k_{34} = \mathcal{D}C\pi(m^4-m^2-12\nu m^2+18\nu+18)\left(\frac{a^3-b^3}{3}\right)}$$

$$k_{41} = D\int_0^{2\pi}\int_b^a \left(-\frac{6\nu m^2\alpha^2}{r} + \frac{m^2\alpha^2}{r}(m^2-3) - \frac{4m^2\beta^2}{r}(1-\nu)\right) r\,dr\,d\theta$$

$$= DC\pi(m^4-7m^2-2\nu m^2)(a-b)$$

$$= k_{14}$$

$$\boxed{k_{41} = k_{14}}$$

$$k_{42} = D\int_0^{2\pi}\int_b^a \left(-6\nu\alpha^2(m^2-1)+\alpha^2(m^2-1)(m^2-3)\right) r\,dr\,d\theta$$

$$= DC\pi(m^4-4m^2-6\nu m^2+6\nu+3)\left(\frac{a^2-b^2}{2}\right)$$

$$= k_{24}$$

$$\boxed{k_{42} = k_{24}}$$

$$k_{43} = D\int_0^{2\pi}\int_b^a \left(12\alpha^2 r-2\nu\alpha^2 r(m^2-3)-6\nu\alpha^2 r(m^2-2)+\alpha^2 r(m^2-2)(m^2-3)\right.$$

$$\left. + 4m^2\beta^2 r(1-\nu)\right) r\,dr\,d\theta$$

$$= DC\pi(m^4-m^2-12\nu m^2+18\nu+18)\left(\frac{a^3-b^3}{3}\right)$$

$$= k_{34}$$

$$\boxed{k_{43} = k_{34}}$$

$$k_{44} = D\int_0^{2\pi}\int_b^a \left(36\alpha^2 r^2-6\alpha^2 r^2\nu(m^2-3)-6\nu\alpha^2 r^2(m^2-3)+\alpha^2 r^2(m^2-3)^2\right.$$

$$\left. + 8m^2\beta^2 r^2(1-\nu)\right) r\,dr\,d\theta$$

$$= DC\pi(36-6\nu m^2+18\nu-6\nu m^2+18\nu+m^4-6m^2+9+8m^2-8\nu m^2)\left(\frac{a^4-b^4}{4}\right)$$

$$\boxed{k_{44} = DC\pi(m^4+2m^2-20\nu m^2+36\nu+45)\left(\frac{a^4-b^4}{4}\right)}$$

2.2.2 Mass Matrix for Isotropic Plates

The mass matrix is derived from the given displacement function. For classical bending of circular plates the ~~poten-tial~~ Kinetic energy for an element in polar coordinates is given by [2],

$$T = \frac{\rho h}{2} \int_0^{2\pi} \int_b^a \dot{w}^2 \, r \, dr \, d\theta$$

where ρ is the mass density.

Equation (2.8) gives the ddisplacement function as

$$w(r,\theta) = \lfloor s \rfloor \, [B] \{\overline{w}\} \cos m\theta$$

$$\dot{w} = \lfloor s \rfloor \, [B] \{\dot{\overline{w}}\} \cos m\theta$$

$$T = \frac{\rho h}{2} \int_0^{2\pi} \int_b^a (\lfloor \dot{\overline{w}} \rfloor [B]^T \{S\} \lfloor S \rfloor [B] \{\dot{\overline{w}}\} \cos^2 m\theta) r \, dr d\theta$$

$$T = \frac{\rho h}{2} \lfloor \dot{\overline{w}} \rfloor [B]^T [\int_0^{2\pi} \int_b^a \{S\} \lfloor S \rfloor \cos^2 m\theta \, r \, dr d\theta][B] \{\dot{\overline{w}}\}$$

Mass matrix $[M] = \rho h [B]^T [\int_0^{2\pi} \int_b^a \{S\} \lfloor S \rfloor \cos^2 m\theta \, r \, dr d\theta][B]$

$$[M] = [B]^T [m] [B]$$

where

$$[m] = \rho h \int_0^{2\pi} \int_b^a \{S\} \lfloor S \rfloor \cos^2 m\theta \, r dr d\theta$$

$$= \rho h \int_0^{2\pi} \int_b^a \begin{Bmatrix} 1 \\ r \\ r^2 \\ r^3 \end{Bmatrix} \lfloor 1 \quad r \quad r^2 \quad r^3 \rfloor \cos^2 m\theta \; r dr d\theta$$

$$= \rho h \int_0^{2\pi} \cos^2 m\theta \, d\theta \int_b^a \begin{bmatrix} r & r^2 & r^3 & r^4 \\ r^2 & r^3 & r^4 & r^5 \\ r^3 & r^4 & r^5 & r^6 \\ r^4 & r^5 & r^6 & r^7 \end{bmatrix} dr$$

$$[m] = C\pi \rho h \begin{bmatrix} \frac{(a^2-b^2)}{2} & (\frac{a^3-b^3}{3}) & (\frac{a^4-b^4}{4}) & (\frac{a^5-b^5}{5}) \\ & (\frac{a^4-b^4}{4}) & (\frac{a^5-b^5}{5}) & (\frac{a^6-b^6}{6}) \\ \text{SYMM.} & & (\frac{a^6-b^6}{6}) & (\frac{a^7-b^7}{7}) \\ & & & (\frac{a^8-b^8}{8}) \end{bmatrix}$$

2.2.3 Determination of Natural Frequencies

It should be noted that the element stiffness matrix [k] and element mass matrix [m] are independent of global boundary conditions. Conversions of [k] and [m] into the total stiffness matrix [K] and total mass matrix [M] of the plate is a simple mechanical operation of addition. All the relevant boundary

conditions are introduced by suppressing the corresponding rows and columns of [K] and [M]. Then the governing equation of motion for undamped free vibrations of multidegree freedom system can be derived as

$$[M] \{\ddot{w}\} + [K] \{w\} = 0 \tag{2.10}$$

For a periodic motion the time coordinate is separated by assuming the displacement vector as

$$\{w\} = \{w_o\} e^{i\omega t} \tag{2.11}$$

where $\{w_o\}$ is the amplitude vector, ω is the natural frequency and 't' is the time coordinate. Using equation (2.11) in equation (2.10), we can obtain the linear homogeneous equation of motion as

$$([K]-[M]\omega^2) \{w\} = 0 \tag{2.12}$$

This is a linear eigenvalue problem as none of the two matrices [K] and [M] involve ω. Both [K] and [M] are real symmetric and positive definite matrices implying real and positive eigenvalues, which in our case are the natural frequencies.

Non-dimensional frequency parameters are calculated using a computer programme in FORTRAN language. IMSL subroutine EIGZF is used to get all the natural frequencies

of the homogeneous equation of motion $(K-M\omega^2) = 0$. These frequencies are then converted into nondimensional form. For this, in the case of isotropic plate the frequency is multiplied by $a^2 (\rho h/D)^{0.5}$ and for orthotropic plate it is multiplied by $a^2(\rho h/D_r)^{0.5}$.

The results and discussion are given in Chapter 4.

CHAPTER 3

GENERAL FORMULATION OF VIBRATION PROBLEM OF CIRCULAR AND ANNULAR POLAR ORTHOTROPIC PLATES

3.1 INTRODUCTION

There are quite a few investigations, reported in the literature, on vibration of polar orthotropic circular [9-14] and annular plates[16-18].Most of these investigations are, however, limited mainly to axisymmetric modes, for particular set of edge conditions, and in some of them, for some specific values of material constants. In one of these investigations Ramaiah et al [18] have obtained natural frequencies of higher modes of vibration based on an assumption that the radial bending moment is zero at a nodal circle. But in this analysis, the presence of a free edge yield inaccurate estimates of frequencies of modes with one or two nodal circles. The existance of singularity at the centre of a polar orthotropic circular disc has been treated in different ways by different investigators. Carrier et al [19] have considered an isotropic homogeneous core extending upto 0.1 times the outside radius at the centre while Sinha et al [20] have taken the size of core to be 0.01 times outside radius. In the present analysis the question of material singularity does not arise because here the frequency is determined by considering the disc as an annular plate having radius ratio 0.001. Here, a similar

annular element formulated in Section 2.2 has been used to determine the frequencies of orthotropic plates.

3.2 STIFFNESS MATRIX FOR ORTHOTROPIC PLATES

For classical bending of orthotropic plates, the strain energy, for an element in polar coordinate is given by [ ]

$$U_1 = \frac{1}{2}\int_0^{2\pi}\int_b^a \Big[D_r\Big(\frac{\partial^2 w}{\partial r^2}\Big)^2 + D_\theta\Big(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\Big)^2 + 2\nu_\theta D_r\frac{\partial^2 w}{\partial r^2}\Big(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\Big) + 2D_{r\theta}\Big(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\Big)^2\Big] r\,dr\,d\theta \tag{3.1}$$

which can be written as

$$U_1 = \frac{1}{2}\int_0^{2\pi}\int_b^a \lfloor d\rfloor\,[V_1]\,\{d\}\; r\,dr\,d\theta \tag{3.2}$$

where $$\lfloor d\rfloor = \Big\lfloor\Big(\frac{\partial^2 w}{\partial r^2}\Big)\;\Big(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\Big)\Big(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\Big)\Big\rfloor \tag{3.3}$$

and

$$[V_1] = \begin{bmatrix} D_r & \nu_\theta D_r & 0 \\ \nu_\theta D_r & D_\theta & 0 \\ 0 & 0 & 2D_{r\theta} \end{bmatrix} \tag{3.4}$$

Assuming a deflection function $w(r,\theta) = (a_1+a_2r+a_3r^2+a_4r^3)\cos m\theta$ and proceding with the analysis given in Section 2.2.1, the element stiffness matrix for orthotropic plate can be obtained as

$$[k] = \int_0^{2\pi}\int_b^a [F]^T\,[V_1]\,[F]\; r\,dr\,d\theta$$

$$[k] = \int_0^{2\pi}\int_b^a \begin{bmatrix} 0 & \frac{m^2\alpha}{r^2} & \frac{m\beta}{r^2} \\ 0 & \frac{\alpha}{r}(m^2-1) & 0 \\ -2\alpha & \alpha(m^2-2) & -m\beta \\ -6\alpha r & \alpha r(m^2-3) & -2m\beta r \end{bmatrix} \begin{bmatrix} D_r & \gamma_\theta D_r & 0 \\ \gamma_\theta D_r & D_\theta & 0 \\ 0 & 0 & 2D_{r\theta} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -2\alpha & -6\alpha r \\ \frac{m^2\alpha}{r^2} & \frac{\alpha}{r}(m^2-1) & \alpha(m^2-2) & \alpha r(m^2-3) \\ \frac{m\beta}{r^2} & 0 & -m\beta & -2m\beta r \end{bmatrix} r\,dr\,d\theta$$

$$[k] = \int_0^{2\pi}\int_b^a$$

$$\begin{bmatrix} \frac{\gamma_\theta D_r m^2\alpha}{r^2} & D_\theta \frac{m^2\alpha}{r^2} & \frac{2D_{r\theta}m\beta}{r^2} \\ \gamma_\theta D_r \frac{\alpha}{r}(m^2-1) & D_\theta \frac{\alpha}{r}(m^2-1) & 0 \\ -D_r 2\alpha+\gamma_\theta D_r\alpha(m^2-2) & -2\gamma_\theta D_r\alpha+D_\theta\alpha(m^2-2) & -2D_{r\theta}m\beta \\ -6D_r\alpha r+\gamma_\theta D_r\alpha r(m^2-3) & -6\gamma_\theta D_r\alpha r+D_\theta\alpha r(m^2-3) & -4D_{r\theta}m\beta r \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -2\alpha & -6\alpha r \\ \frac{m^2\alpha}{r^2} & \frac{\alpha}{r}(m^2-1) & \alpha(m^2-2) & \alpha r(m^2-3) \\ \frac{m\beta}{r^2} & 0 & -m\beta & -2m\beta r \end{bmatrix} r\,dr\,d\theta$$

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

where,
$$k_{11} = \int_0^{2\pi} \int_b^a \left(D_\theta \frac{m^4\alpha^2}{r^4} + \frac{2D_{r\theta}}{r^4} m^2\beta^2 \right) rdrd\theta$$

$$= C\pi \; (D_\theta m^4 + 2D_{r\theta} m^2)\left(\frac{a^2-b^2}{2(ab)^2}\right)$$

$$\boxed{k_{11} = C\pi(m^4 D_\theta + 2m^2 D_{r\theta})\left(\frac{a^2-b^2}{2(ab)^2}\right)}$$

$$k_{12} = \int_0^{2\pi} \int_b^a \left[D_\theta \frac{m^2\alpha}{r^3} (m^2-1)\right] rdrd\theta$$

$$\boxed{k_{12} = C\pi(D_\theta(m^4-m^2))\left(\frac{a-b}{ab}\right)}$$

$$k_{13} = \int_0^{2\pi} \int_b^a \left(- \frac{2\; {}_\theta D_r m^2\alpha^2}{r^2} + \frac{D_\theta \alpha^2 (m^4-2m^2)}{r^2} - \frac{2D_{r\theta} m^2\beta^2}{r^2} \right) rdrd\theta$$

$$\boxed{k_{13} = C\pi(-2\; {}_\theta D_r m^2 + D_\theta(m^4-2m^2) - 2D_{r\theta} m^2) \; \ln(a/b)}$$

$$k_{14} = \int_{o}^{2\pi} \int_{b}^{a} \left(- \frac{6\nu_{\theta} D_r m^2 \alpha^2}{r} + \frac{D_{\theta} m^2 \alpha^2 (m^2-3)}{r} - \frac{4D_{r\theta} m^2 \beta^2}{r}\right) r dr d\theta$$

$$\boxed{k_{14} = C\pi(-6\nu_{\theta} D_r m^2 + D_{\theta}(m^4-3m^2) - 4D_{r\theta} m^2)(a-b)}$$

$$k_{21} = \int_{o}^{2\pi} \int_{b}^{a} (D_{\theta} \frac{m^2 \alpha}{r^3} (m^2-1) r dr d\theta$$

$$= C\pi(D_{\theta}(m^4-m^2) \left(\frac{a-b}{ab}\right)$$

$$= k_{12}$$

$$\boxed{k_{21} = k_{12}}$$

$$k_{22} = \int_{o}^{2\pi} \int_{b}^{a} D_{\theta} \frac{\alpha^2}{r^2} (m^2-1)^2 \; r dr d\theta$$

$$\boxed{k_{22} = C\pi \; D_{\theta}(m^2-1)^2 \; \ln(a/b)}$$

$$k_{23} = \int_{o}^{2\pi} \int_{b}^{a} (-2\nu_{\theta} D_r \frac{\alpha^2}{r} (m^2-1) + D_{\theta} \frac{\alpha^2}{r} (m^2-1)(m^2-2)) r dr d\theta$$

$$\boxed{k_{23} = C\pi(-2\nu_{\theta} D_r (m^2-1) + D_{\theta}(m^2-1)(m^2-2))(a-b)}$$

$$k_{24} = \int_{o}^{2\pi} \int_{b}^{a} (-6\nu_{\theta} D_r \alpha^2 (m^2-1) + D_{\theta} \alpha^2 (m^2-1)(m^2-3)) r dr d\theta$$

$$\boxed{k_{24} = C\pi(-6\nu_{\theta} B_r (m^2-1) + D_{\theta}(m^4-4m^2+3))\left(\frac{a^2-b^2}{2}\right)}$$

$$k_{31} = \int_o^{2\pi}\int_b^{a} \left(-\frac{2\gamma_{\theta}D_r m^2\alpha^2}{r^2} + \frac{D_\theta\alpha^2(m^4-2m^2)}{r^2} - \frac{2D_{r\theta}m^2\beta^2}{r^2}\right) r\,dr\,d\theta$$

$$= C\pi(-2\gamma_{\theta}D_r m^2+D_\theta(m^4-2m^2)-2D_{r\theta}m^2)\ \ln(a/b)$$

$$= k_{13}$$

$$\boxed{k_{31} = k_{13}}$$

$$k_{32} = \int_o^{2\pi}\int_b^{a} (-2\gamma_{\theta}D_r \frac{\alpha^2}{r}(m^2-1)+ D_\theta \frac{\alpha^2}{r}(m^2-1)(m^2-2)r\,dr\,d\theta$$

$$= C\pi(-2\gamma_{\theta}D_r(m^2-1)+D_\theta(m^2-1)(m^2-2)(a-b)$$

$$= k_{23}$$

$$\boxed{k_{32} = k_{23}}$$

$$k_{33} = \int_o^{2\pi}\int_b^{a} (4D_r\alpha^2-2\gamma_{\theta}D_r\alpha^2(m^2-2)-2\gamma_{\theta}D_r\alpha^2(m^2-2)$$

$$+ D_\theta\alpha^2(m^2-2)^2+2D_{r\theta}m^2\beta^2)r\,dr\,d\theta$$

$$\boxed{k_{33} = C\pi(4D_r-4\gamma_{\theta}D_r(m^2-2)+D_\theta(m^2-2)^2+2D_{r\theta}m^2)\left(\frac{a^2-b^2}{2}\right)}$$

$$k_{34} = \int_{o}^{2\pi}\int_{b}^{a} (12D_r\alpha^2 r - 6\nu_\theta D_r\alpha^2 r(m^2-2) - 2\nu_\theta D_r\alpha^2 r(m^2-3)$$

$$+ D_\theta\alpha^2 r(m^2-3)(m^2-2) + 4D_{r\theta}m^2\beta^2 r)\,rdrd\theta$$

$$= C\pi(12D_r - 6\nu_\theta D_r(m^2-2) - 2\nu_\theta D_r(m^2-3) + D_\theta(m^2-3)(m^2-2)$$

$$+4D_{r\theta}m^2)\left(\frac{a^3-b^3}{3}\right)$$

$$\boxed{k_{34} = C\pi(12D_r - \nu_\theta D_r(8m^2-18) + D_\theta(m^4-5m^2+6) + 4D_{r\theta}m^2)\left(\frac{a^3-b^3}{3}\right)}$$

$$k_{41} = \int_{o}^{2\pi}\int_{b}^{a} \left(-\frac{6\nu_\theta D_r m^2\alpha^2}{r} + \frac{D_\theta m^2\alpha^2(m^3-3)}{r} - \frac{4D_{r\theta}m^2\beta^2}{r}\right)rdrd\theta$$

$$= C\pi(-6\nu_\theta D_r m^2 + D_\theta(m^4-3m^2) - 4D_{r\theta}m^2)(a-b)$$

$$= k_{14}$$

$$\boxed{k_{41} = k_{14}}$$

$$k_{42} = \int_{o}^{2\pi}\int_{b}^{a} (-6\nu_\theta D_r\alpha^2(m^2-1) + D_\theta\alpha^2(m^2-1)(m^2-3))\,rdrd\theta$$

$$= C\pi(-6\nu_\theta D_r(m^2-1) + D_\theta(m^4-4m^2+3))\left(\frac{a^2-b^2}{2}\right)$$

$$= k_{24}$$

$$\boxed{k_{42} = k_{24}}$$

$$k_{43} = \int_{o}^{2\pi}\int_{b}^{a} (12D_r\alpha^2 r - 6\nu_\theta D_r\alpha^2 r(m^2-2) - 2\nu_\theta D_r\alpha^2 r(m^2-3)$$

$$+ D_\theta\alpha^2 r(m^2-3)(m^2-2) + 4D_{r\theta}m^2\beta^2 r)\, r\,dr\,d\theta$$

$$= C\pi(12D_r - \nu_\theta D_r(8m^2-18) + D_\theta(m^4-5m^2+6) + 4D_{r\theta}m^2)\left(\frac{a^3-b^3}{3}\right)$$

$$= k_{34}$$

$$\boxed{k_{43} = k_{34}}$$

$$k_{44} = \int_{o}^{2\pi}\int_{b}^{a} (36D_r\alpha^2 r^2 - 6\nu_\theta D_r\alpha^2 r^2(m^2-3) - 6\nu_\theta D_r\alpha^2 r^2(m^2-3)$$

$$+ D_\theta\alpha^2 r^2(m^2-3)^2 + 8D_{r\theta}m^2\beta^2 r^2)\, r\,dr\,d\theta$$

$$\boxed{k_{44} = (36D_r - 12\nu_\theta D_r(m^2-3) + D_\theta(m^2-3)^2 + 8D_{r\theta}m^2)\left(\frac{a^4-b^4}{4}\right)}$$

3.3 MASS MATRIX FOR ORTHOTROPIC PLATES

Since the mass matrix is independent of material properties, D_θ, D_r and $D_{r\theta}$, the expression for mass matrix of polar orthotropic plate will be same as that for isotropic plate as given in Section 2.2.2.

Determination of natural frequency similarly follows as given in Section 2.2.3 and the results and discussion are given in Chapter 4.

CHAPTER 4

RESULTS AND DISCUSSION

4.1 ISOTROPIC PLATES

4.1.1 Isotropic Annular Plates

The analysis presented in Section 2.2 has been applied for all nine possible combinations of clamped, Free and simply supported annular plates. Nondimensional frequency parameters are obtained for all the cases and for different ratio of radii (b/a). For the three cases of free inner edge, the values are tabulated (Tables 4.1, 4.2 and 4.3) and for all the cases graphs are given in Figs. 4.1 to 4.9 . The accuracy and convergence of the method is checked by determining natural frequencies with one, two, four and eight elements. The results are compared with the exact values [3]. From the graphs it can be seen that as (b/a) approaches zero each of the curves tends to a finite limit in all but two cases. The two exceptions are for the mode characterised by one nodal diameter for the cases of a plate with a free outside edge and either clamped or simply supported inner edge. The limiting frequency for each of these modes is zero since the motion is simply a rigid-body rotation of the plate about a diameter. It can also be seen that a variety of relationship between natural frequency and ratio of radii are possible. These range from the situation

Table 4.1

Comparison of estimates of frequency parameter $[\omega a^2(\rho h/D)^{0.5}]$ with exact values; S-F isotropic annularplate; $\nu = 0.3$

n	m	NE	0.1	0.2	0.3	0.4	0.5
		1	4.91	4.78	4.71	4.79	5.09
		2	4.87	4.73	4.67	4.77	5.08
0	0	4	4.86	4.72	4.67	4.76	5.08
		8	4.86	4.72	4.66	4.77	5.07
		Exact	4.86	-	4.66	-	5.07
		1	14.66	14.07	13.82	13.66	13.51
		2	14.24	14.13	13.87	13.27	12.66
0	1	4	13.97	13.90	13.40	12.52	11.96
		8	13.91	13.71	12.99	12.15	11.71
		Exact	13.88	-	12.82	-	11.62
		1	29.30	27.42	25.45	23.88	23.01
		2	26.75	25.76	24.77	23.79	22.99
0	2	4	25.67	25.06	24.33	23.43	22.65
		8	25.49	24.81	24.00	23.01	22.29
		Exact	25.45	-	24.12	-	22.31
		1	33.43	34.65	41.08	53.20	74.43
		2	24.82	31.61	37.20	47.66	66.10
1	0	4	29.52	31.44	37.06	47.49	65.77
		8	29.46	31.41	37.05	47.47	65.45
		Exact	29.41	-	37.01	-	65.76
		1	60.28	65.19	71.51	79.41	94.30
		2	67.60	60.62	58.49	64.15	81.58
1	1	4	51.36	49.80	49.31	55.58	72.17
		8	49.07	47.19	47.00	53.84	70.07
		Exact	48.08	-	45.77	-	69.89
		1	97.80	78.08	75.05	80.92	97.08
		2	81.05	77.94	73.71	76.25	90.76
1	2	4	73.76	70.61	67.86	70.20	83.26
		8	70.23	68.23	66.41	68.04	80.90
		Exact	69.23	-	65.10	-	81.13

Table 4.2

Comparison of estimates of frequency parameters $[\omega a^2(\rho h/D)^{0.5}]$ with exact values. C-F isotropic annular plate ; $\nu = 0.3$

n	m	NE	0.1	0.2	0.3	0.4	0.5
		1	10.27	10.52	11.50	13.65	17.76
		2	10.21	10.44	11.44	13.61	17.72
0	0	4	10.17	10.41	11.43	13.60	17.72
		8	10.16	10.41	11.42	13.60	17.72
		Exact	10.16	-	11.42	-	17.68
		1	23.66	21.68	21.07	21.27	23.06
		2	21.97	21.63	21.21	21.06	22.79
0	1	4	21.45	21.28	20.53	20.26	22.31
		8	21.30	20.89	19.87	19.82	22.14
		Exact	21.15	-	19.50	-	21.98
		1	62.52	44.41	36.77	33.30	33.04
		2	38.26	36.05	34.10	32.69	32.94
0	2	4	35.31	34.31	33.16	32.08	32.49
		8	34.77	33.90	32.65	32.09	31.67
		Exact	34.53	-	32.55	-	32.05
		1	51.68	54.01	66.99	90.51	131.27
		2	39.88	43.21	51.95	67.46	94.20
1	0	4	39.65	43.09	51.80	67.27	93.67
		8	39.53	43.03	51.76	67.11	92.86
		Exact	39.49	-	51.72	-	93.85
		1	79.80	78.55	93.15	127.94	201.42
		2	91.21	92.02	89.18	102.72	138.51
1	1	4	66.74	64.14	65.43	77.37	103.97
		8	62.06	59.54	61.37	73.53	98.84
		Exact	59.98	-	59.81	-	97.32
		1	351.56	304.69	348.44	467.26	712.96
		2	176.58	118.77	104.79	112.79	144.00
1	2	4	92.90	87.62	84.51	91.18	113.88
		8	85.58	82.77	81.12	87.71	107.85
		Exact	83.44	-	78.98	-	107.56

Table 4.3

Comparison of estimates of frequency parameters$[\omega a^2(\rho h/D)^{0.5}]$ with exact values; F-F isotropic annular plate; $\lambda = 0.3$

n	m	NE	0.1	0.2	0.3	0.4	0.5
0	2	1	5.31	5.15	4.91	4.61	4.27
		2	5.30	5.15	4.91	4.61	4.27
		4	5.31	5.16	4.90	4.62	4.17
		8	5.29	4.95	4.81	2.73	4.00
		Exact	5.30	-	4.91	-	4.08
1	0	1	8.86	8.53	8.40	8.64	9.32
		2	8.83	8.48	8.37	8.62	9.32
		4	8.79	8.45	8.36	8.61	9.32
		8	8.78	8.44	8.36	8.61	9.30
		Exact	8.77	-	8.36	-	9.31
1	1	1	21.43	20.51	20.32	20.31	20.49
		2	20.67	20.54	20.13	19.32	18.86
		4	20.59	20.42	19.41	18.14	17.76
		8	20.50	20.04	18.76	17.61	17.15
		Exact	20.49	-	18.34	-	17.18
2	0	1	44.39	46.95	57.36	76.04	108.29
		2	38.49	41.82	50.47	65.84	92.46
		4	38.37	41.78	50.41	65.77	91.96
		8	38.25	41.71	50.35	65.69	91.71
		Exact	38.17	-	50.30	-	91.36
2	1	1	92.14	102.23	119.11	133.17	155.45
		2	93.24	77.25	74.92	85.45	113.22
		4	62.96	61.16	62.53	73.83	99.17
		8	60.52	58.18	59.99	71.86	97.15
		Exact	58.99	-	58.82	-	96.33

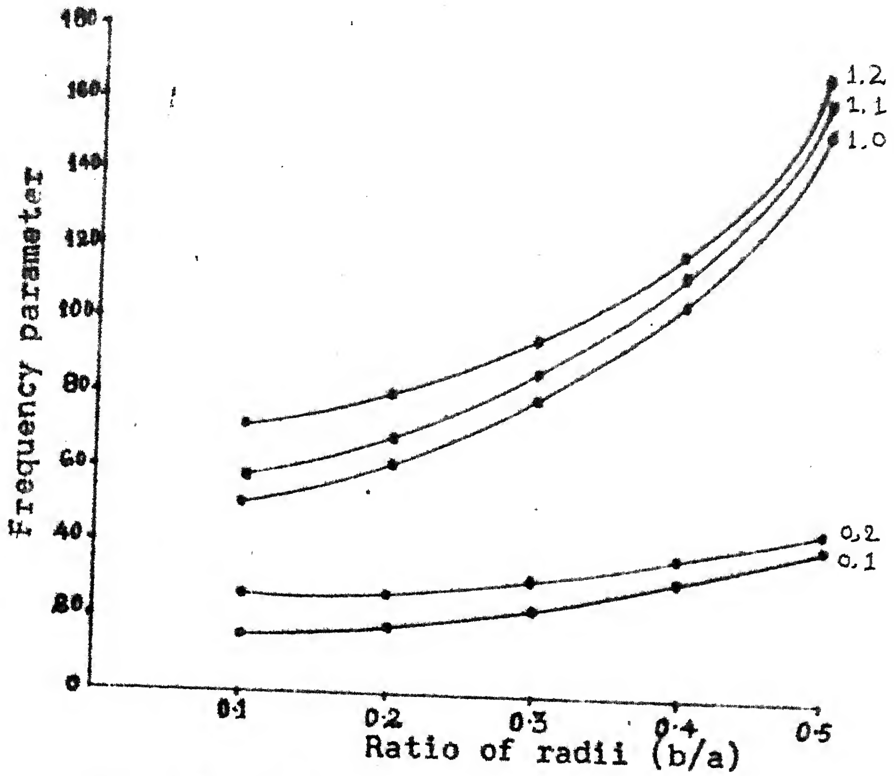

Fig. 4.1 Nondimensional frequency parameter Vs ratio of radii for S-S plate

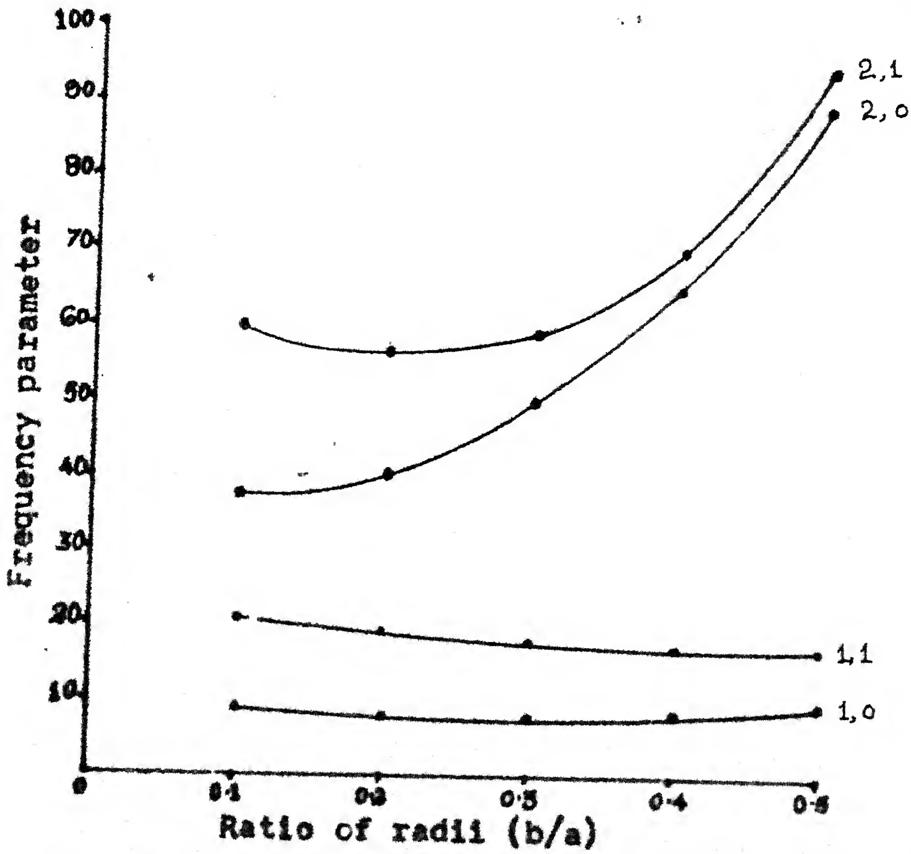

Fig. 4.2 Nondimensional frequency parameter Vs ratio of radii for F-F plate

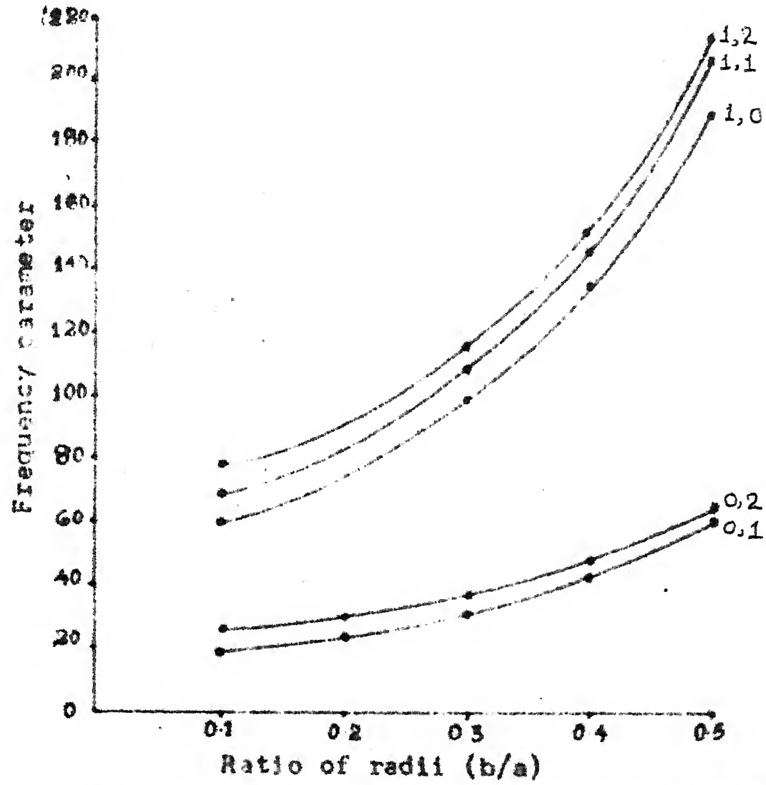

Fig. 4.3 Nondimensional frequency parameter Vs ratio of radii for S-C plate

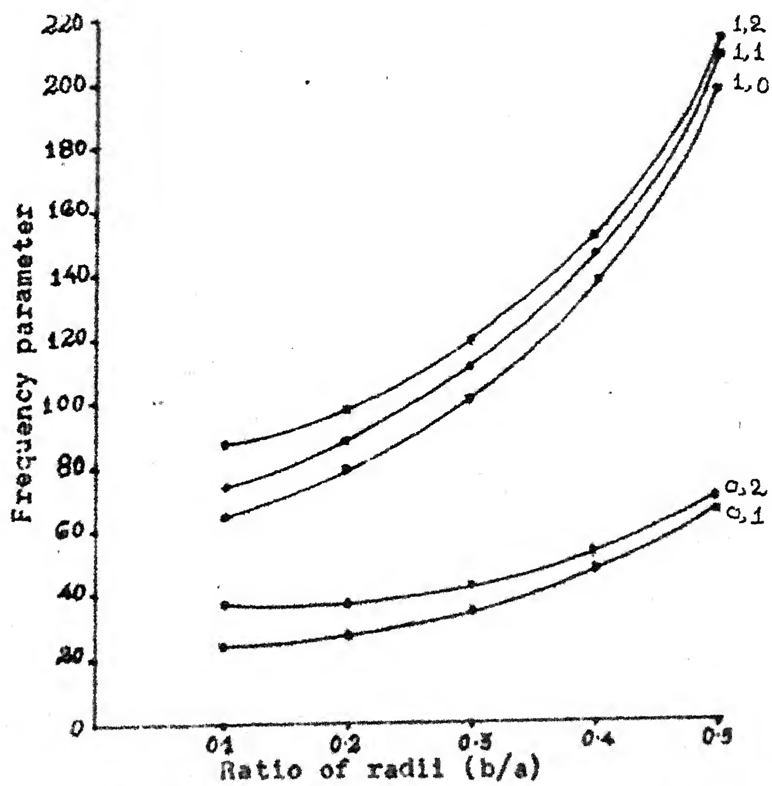

Fig. 4.4 Nondimensional frequency parameter Vs ratio of radii for C-S plate

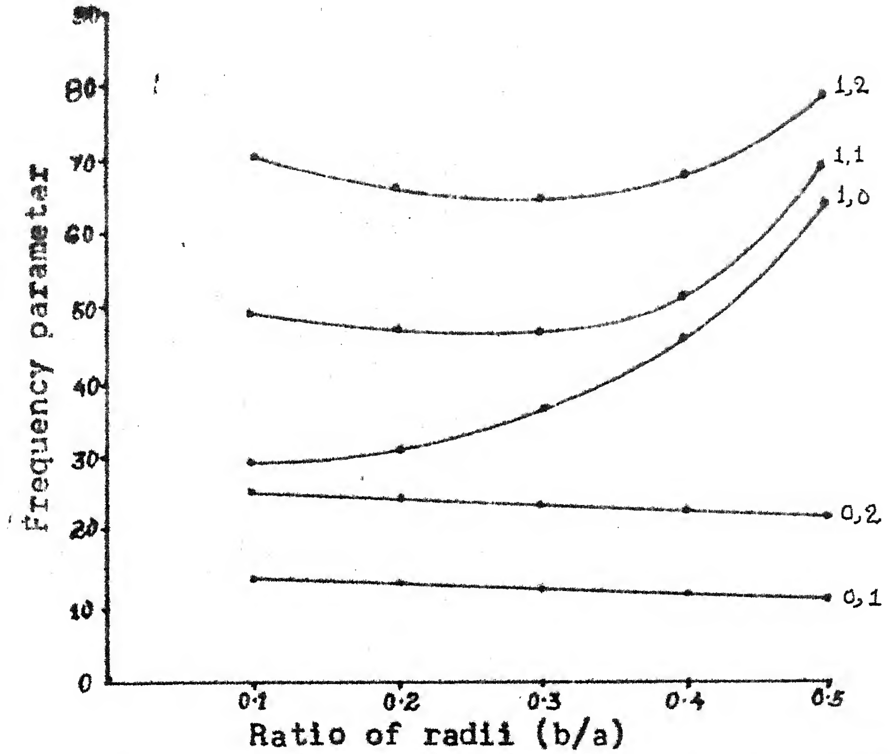

Fig. 4.5 Nondimensional frequency parameter Vs ratio of radii for S-F plate

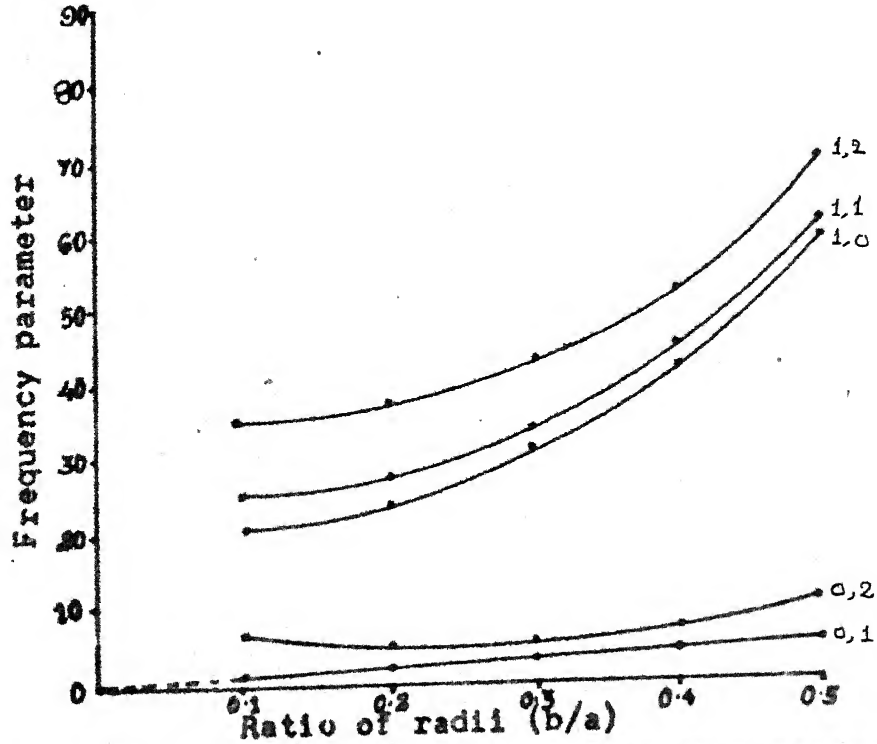

Fig. 4.6 Nondimensional frequency parameter Vs ratio of radii for F-S plate

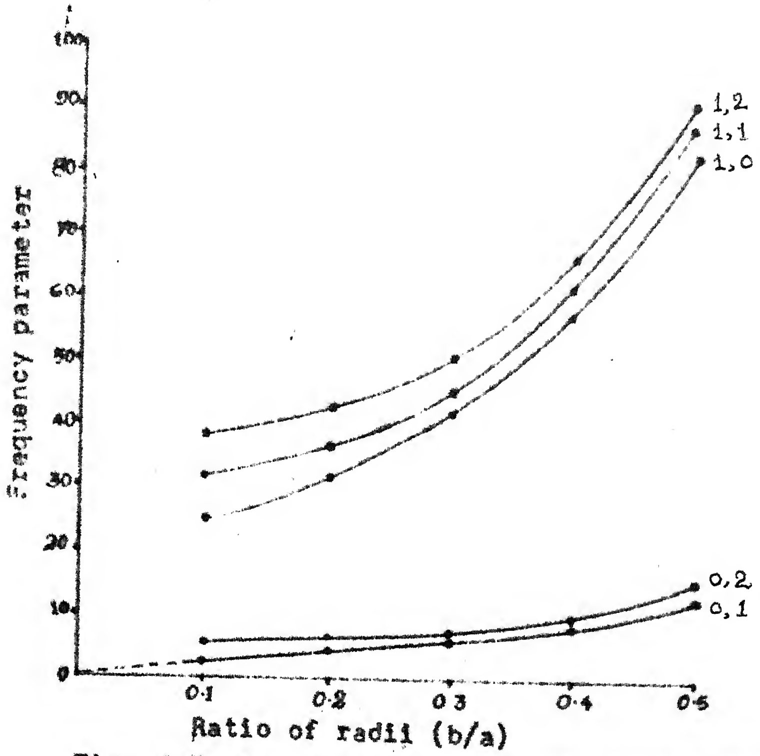

Fig. 4.7 Nondimensional frequency parameter Vs ratio of radii for F-C plate

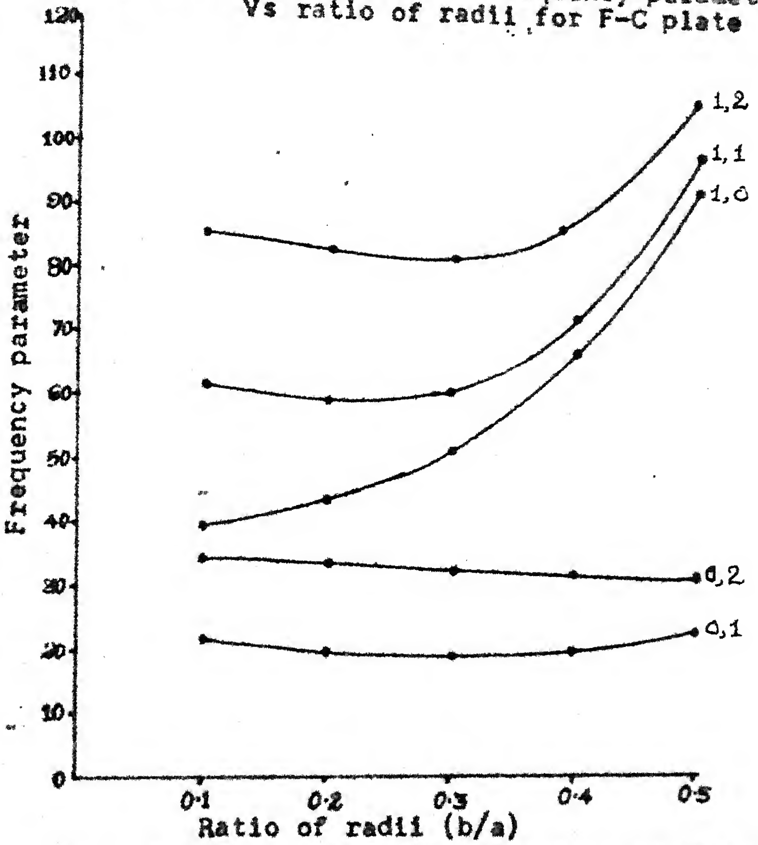

Fig. 4.8 Nondimensional frequency parameter Vs ratio of radii for C-F plate

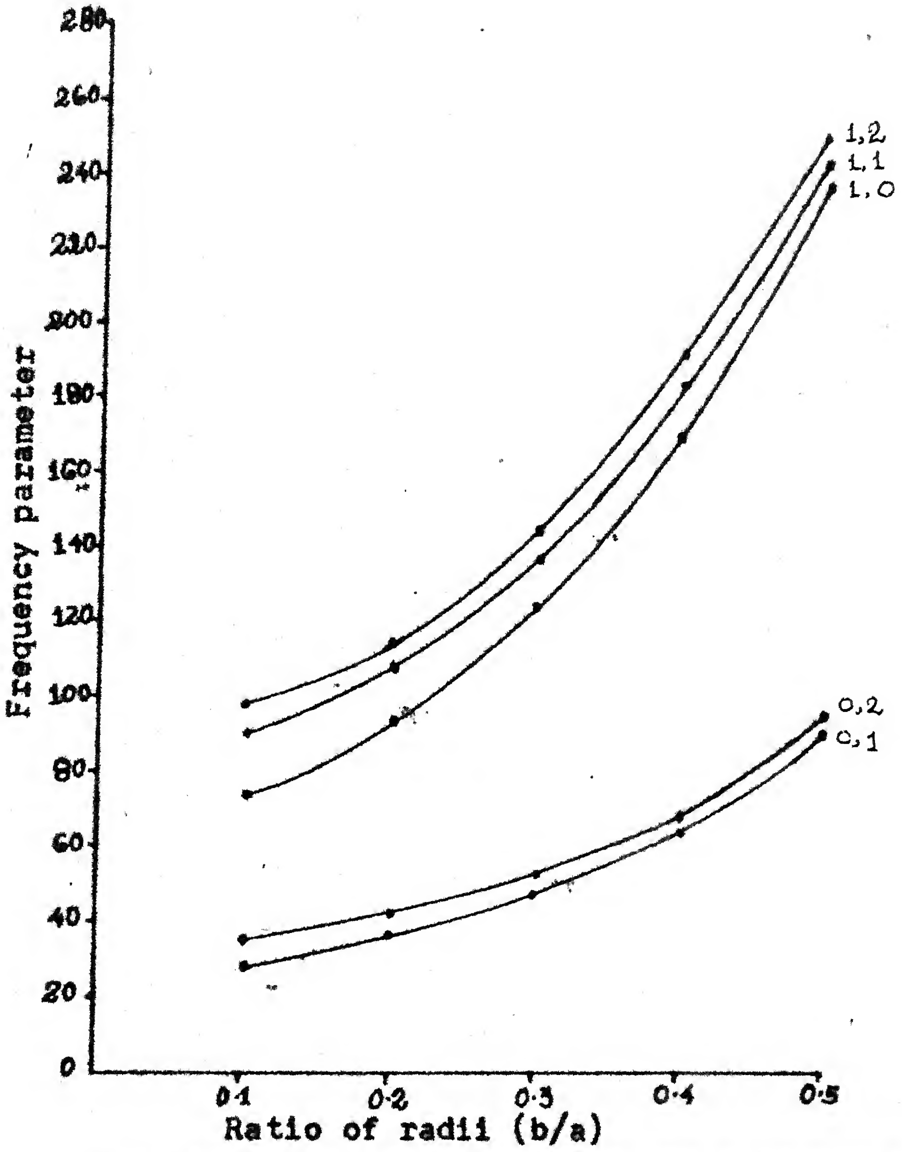

Fig. 4.9 Nondimensional frequency parameter Vs ratio of radii for C–C plate

where (a) frequency constantly increases, (b) frequency initially decreases and then increases, and (c) frequency continuously decreases as ratio (b/a) increases.

In all the cases it can be seen that the mode of vibration with one nodal ~~diameter~~ circle and zero nodal ~~circle~~ diameter is the fourth mode of vibration and that with zero nodal ~~diameter~~ circle and three nodal ~~circles~~ diameter is of higher order. Hence, in Table 4.4 the lowest four mode of vibrations are tabulated for all the cases with ratio of radii (b/a) = 0.1 and are compared with exact values. It can be seen from the table that for higher mode of vibration the accuracy decreases. It is because of the fact that in the analysis the effect of rotary inertia and additional deflection caused by shear forces are not taken into account.

4.1.2 Isotropic Circular Plates

When the ratio of radii (b/a) is reduced to 0.001, the presence of hole is seen to have very small effect on the natural frequencies. Hence, to determine the natural frequencies of circular plates, annular element is used and the ratio of radii is taken to be 0.001. Natural frequencies are obtained for all the three cases of clamped, free and simply supported outer outer edge conditions. The results are compared with exact values [3] in Table 4.5. In order to compare the efficiency of using annular elements, the values obtained for a free plate is compared with that obtained by using a 3x12 grid sector element with 55 degrees of freedom [7] which is also

Table 4.4

Comparison of estimates of frequency parameters $[\omega a^2(\rho h/D)^{0.5}]$ with exact values. Isotropic annular plate : $\nu = 0.3, (a/b) = 0.1$

n	m	Edge conditions								
		S–F	F–S	C–F	F–C	C–S	S–C	C–C	S–S	F–F
0	0	4.86 (4.86)	3.45 (3.45)	10.16 (10.16)	4.24 (4.23)	22.70 (22.64)	17.81 (17.85)	27.31 (27.26)	14.49 (14.54)	-
0	1	13.91 (13.88)	2.87 (2.30)	21.30 (21.15)	3.33 (3.14)	26.72 (25.12)	19.21 (19.00)	29.11 (28.42)	17.80 (16.69)	-
0	2	25.44 (25.45)	5.30 (5.42)	34.77 (34.53)	5.82 (5.62)	35.80 (35.36)	27.65 (26.77)	38.21 (36.68)	26.11 (25.94)	5.29 (5.30)
1	0	29.46 (29.41)	20.89 (20.82)	39.53 (39.49)	25.29 (25.28)	65.61 (65.60)	60.21 (60.14)	75.43 (75.34)	51.76 (51.72)	8.78 (8.77)

Values in parentheses are exact values [3]

Table 4.5

Comparison of estimates of frequency parameter $[\omega a^2(\rho h/D)^{0.5}]$ with exact values. Isotropic circular plate : $\nu = 0.3$

m	Simply supported plate FEM	Simply supported plate Exact	Clamped plate FEM	Clamped plate Exact	Free plate FEM	Free plate Exact	Free plate Ref [7]
0	4.94	4.97	10.22	10.24	-	-	-
1	13.92	13.91	21.31	21.25	-	-	-
2	25.71	25.70	35.11	34.80	5.45	5.24	5.90
0	29.70	29.70	39.77	39.80	9.00	9.06	8.98
1	49.26	48.60	62.49	60.80	20.51	20.50	20.24
2	71.44	70.10	87.39	84.60	35.38	35.50	36.01
0	74.21	74.10	89.15	89.10	38.45	38.40	38.12

given in Table 4.5. It can be seen that the values obtained by using annular element is very close to the exact values than the values obtained by using the sector element.

4.2 ORTHOTROPIC PLATES

4.2.1 Polar Orthotropic Annular Plates

Natural frequencies are obtained for all the nine combinations of free, clamped and simply supported edge conditions. Frequencies are calculated for different rigidity ratio and for different hole sizes. In Tables 4.6 to 4.8, the fundamental frequency parameters for different hole sizes and for different rigidity ratios are given for S-F, C-F and F-F annular plates. Table 4.9 gives the nondimensional frequency parameters for all the nine combinations of edge conditions for $b/a = 0.1$. Numerical values are available in ref [17] only for the three cases of free inner edge and for rigidity ratios 2.0. The estimated frequencies for those cases are compared and good agreement is found. [It is to be noted that the values in ref. [17] are only approximate values obtained by Rayleigh-Ritz method]. As a check for the present analysis, the values obtained for rigidity ratio 1.0 is compared with exact values of isotropic plates. In Table 4.10 the frequencies are tabulated for $n = 1$ and $m = 0$ for all the cases of nine combinations of edge conditions. It is seen that the accuracy decreases at higher mode of vibration, and this is because of the ignorance of additional deflection

Table 4.6

Variation of fundamental frequency parameter $[\omega a^2(\rho h/D_r)^{0.5}]$ with rigidity ratio (D_θ/D_r) and ratio of radii (b/a); orthotropic S-F plate ; $D_{r\theta} = 0.35$, $\nu_\theta = 0.3$

Rigidity ratio (D_θ/D_r)	Ratio of radii (b/a)				
	0.1	0.2	0.3	0.4	0.5
0.2	2.29	1.85	1.88	2.34	2.77
0.4	3.35	2.89	3.03	2.82	5.57
0.6	3.93	3.68	3.97	3.61	-
0.8	4.49	4.16	4.22	4.45	5.10
1.0	4.89 (4.86)	4.89	5.00 (4.66)	4.95	5.84 (5.07)
1.2	5.23	5.21	5.04	5.50	6.78
1.4	5.49	5.48	5.30	5.60	8.04
1.6	5.69	5.76	6.08	5.75	7.26
1.8	6.03	6.00	6.28	5.81	7.65
2.0	6.21 6.11*	6.33	6.42	6.53	5.08 7.14*

Values in parentheses are exact values of isotropic plate

*Values from Ref. [17]

Table 4.7

Variation of fundamental frequency parameter$[\omega a^2(\rho h/D_r)^{0.5}]$ with rigidity ratio (D_θ/D_r) and ratio of radii (b/a); orthotropic C-F plate ; $D_{r\theta} = 0.35$; $\nu_\theta = 0.3$

Rigidity ratio (D_θ/D_r)	Ratio of radii (b/a)				
	0.1	0.2	0.3	0.4	0.5
0.2	7.58	7.99	9.68	7.26	10.02
0.4	8.61	8.87	10.18	12.51	-
0.6	9.35	9.52	11.24	12.66	16.54
0.8	9.85	10.03	11.54	13.27	16.93
1.0	10.33 (10.16)	10.53	11.56 (11.42)	13.37	17.91 (17.68)
1.2	10.74	10.99	11.91	13.88	19.42
1.4	11.08	11.38	12.37	14.28	19.72
1.6	11.34	11.80	12.77	15.33	20.80
1.8	11.66	12.04	13.14	14.97	19.76
2.0	11.92 11.45*	12.21	13.45	14.94	19.86 18.62*

Values in parentheses are exact values of ositropic plate

*Values from Ref. [17]

Table 4.8

Variation of frequency parameter $[\omega a^2(\rho h/D_r)^{0.5}]$ with rigidity ratio of radii (b/a). Orthotropic F-F plate; $D_{r\theta} = 0.35$; $\nu_\theta = 0.3$, n = 1, m = 0

Rigidity ratio (D_θ/D_r)	Ratio of radii (b/a)				
	0.1	0.2	0.3	0.4	0.5
0.2	3.93	3.27	3.06	3.07	3.36
0.4	5.90	5.24	5.03	5.10	5.44
0.6	7.16	6.58	6.38	6.51	7.04
0.8	8.11	7.64	7.46	7.66	8.28
1.0	8.90 (8.77)	8.52	8.39 (8.36)	8.64	9.42 (9.31)
1.2	9.57	9.29	9.23	9.52	10.29
1.4	10.17	9.98	9.97	10.31	11.12
1.6	10.71	10.60	10.65	11.06	12.00
1.8	11.21	11.18	11.28	11.73	12.76
2.0	11.68 11.34*	11.71	11.87	12.39	13.48 13.09*

Note : Values in parentheses are exact values of isotropic circular plate

* Values from Ref. [17]

Table 4.9

Variation of frequency parameter $[\omega a^2(\rho h/D_r)]^{0.5}$ with rigidity ratio (D_θ/D_r). Orthotropic plate; $D_{r\theta} = 0.35$, $\nu_\theta = 0.3$, $(b/a) = 0.1$, $n = 0$, $m = 0$

D_θ/D_r	S-F	C-F	F-F**	S-S	C-S	F-S	S-C	C-C	F-C
0.2	2.29	7.58	3.93	11.84	20.37	0.82	18.04	29.22	3.17
0.4	3.35	8.61	5.90	12.69	21.28	2.01	18.38	29.47	3.79
0.6	3.93	9.35	7.16	13.43	21.64	2.84	18.62	29.75	3.94
0.8	4.49	9.85	8.11	13.97	22.46	3.09	18.92	30.05	4.44
1.0	4.89 (4.86)	10.33 (10.16)	8.90 (8.77)	14.53 (14.54)	22.82 (22.64)	3.45 (3.45)	17.92 (17.85)	25.29 (27.26)	4.48 (4.23)
1.2	5.23	10.74	9.57	15.05	23.83	3.82	19.48	30.62	4.71
1.4	5.49	11.08	10.17	15.48	24.32	4.23	19.64	30.80	5.20
1.6	5.69	11.34	10.71	15.97	24.63	4.48	19.90	31.11	5.02
1.8	6.03	11.66	11.21	16.58	25.23	4.81	20.18	31.30	5.19
2.0	6.21 6.11*	11.92 11.45*	11.68 11.34*	17.08	25.80	4.75	20.43	31.64	5.36

*Values from Ref. [17]

Values in parentheses are exact values of isotropic plate

**Values for $n = 1$, $m = 0$

Table 4.10

Variation of frequency parameter with rigidity ratio: Orthotropic plate $D_{r\theta} = 0.35$, $\gamma_{\theta} = 0.3$ $(b/a) = 0.1$, $n = 1$, $m = 0$

D_{θ}/D_r	S–F	C–F	F–F*	S–S	C–S	F–S	S–C	C–C	F–C
0.2	12.42	19.65	5.88	51.23	66.62	17.39	69.07	93.42	25.98
0.4	12.51	19.28	7.90	52.03	67.88	18.57	69.35	93.80	26.46
0.6	12.65	19.09	9.30	52.88	68.72	19.49	69.74	94.22	26.67
0.8	12.80	18.97	10.46	53.69	69.25	19.99	70.04	94.45	27.08
1.0	12.99 (13.88)	18.92 (21.15)	11.48 (12.44)	54.43 (51.72)	69.91 (65.60)	20.94 (20.82)	70.43 (60.14)	94.90 (90.54)	27.57 (27.26)
1.2	13.17	18.89	12.41	55.23	71.16	21.65	70.73	95.16	27.87
1.4	13.35	18.82	13.26	55.92	71.97	22.41	71.08	95.43	28.30
1.6	13.50	18.79	14.06	56.75	72.65	23.19	71.28	95.88	28.61
1.8	13.66	18.73	14.82	57.41	73.15	23.63	71.70	96.19	28.93
2.0	13.82	18.69	15.64	58.10	73.94	24.14	72.00	96.54	29.17

*Value for $n = 0$, $m = 3$

Values in parentheses are exact values of isotropic plate

caused by shear forces in the analysis. It can be seen that the frequency depends on rigidity ratio and hole sizes. The graphs given (Figs. 4.10 - 4.27) shows the variation of frequency with rigidity ratio and variation of frequency with hole size. In all the cases it can be seen that the frequency increases with increase in rigidity ratio and the variation is linear for $D_\theta/D_r > 1$. In some cases, where the variation of frequency with rigidity ratio is small, even in the range $D_\theta/D_r < 1$, the variation is seen to be linear. From Fig.4.17 it can be seen that the rate of increase in frequency with rigidity ratio is more in the range $D_\theta/D_r < 1$ and the rate of increase in frequency decreases with increase in rigidity ratio. Another thing to be noted is the variation of frequency with ratio of radii. In the case of a F-F plate as shown in Fig. 4.15, the frequency initially decreases and then increases with hole size, and the decreasing tendency is found to be more in the range $D_O/D_r < 1$. For a value of rigidity ratio 2.0, there is no decrease in frequency with increase in hole size, but frequency increases continuously with increase in hole size. For all the cases the effect of hole size on frequency depends upon the edge conditions.

4.2.2 Polar Orthotropic Circular Plate

Natural frequencies are obtained for all the three cases of clamped, free and simply supported edge conditions. In order to study the variation of frequency with rigidity ratio,

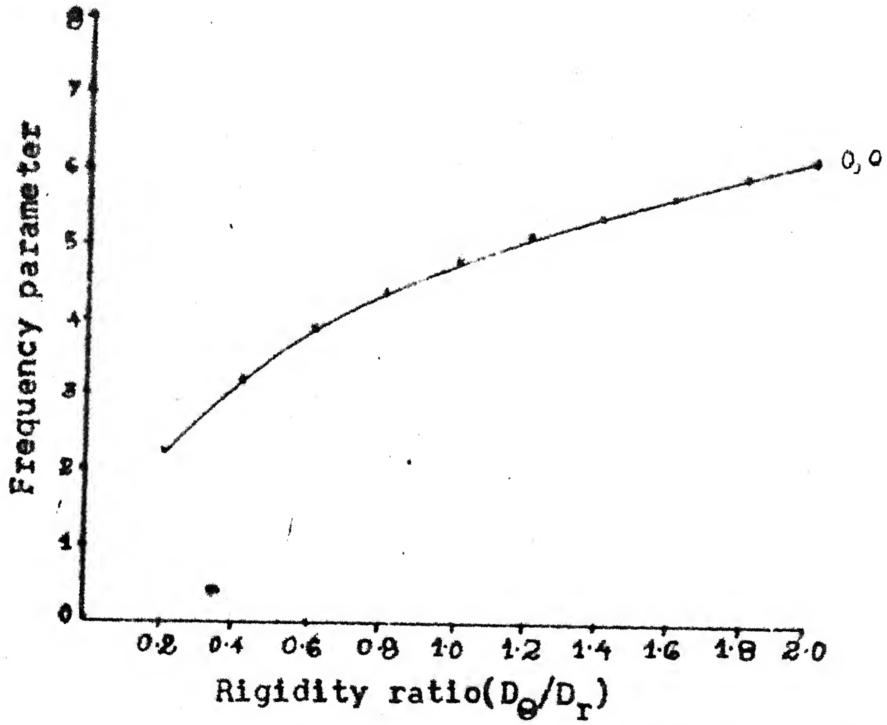

Fig. 4.10 Nondimensional frequency parameter Vs rigidity ratio for S-F polar orthotropic plate; $b/a = 0.1$

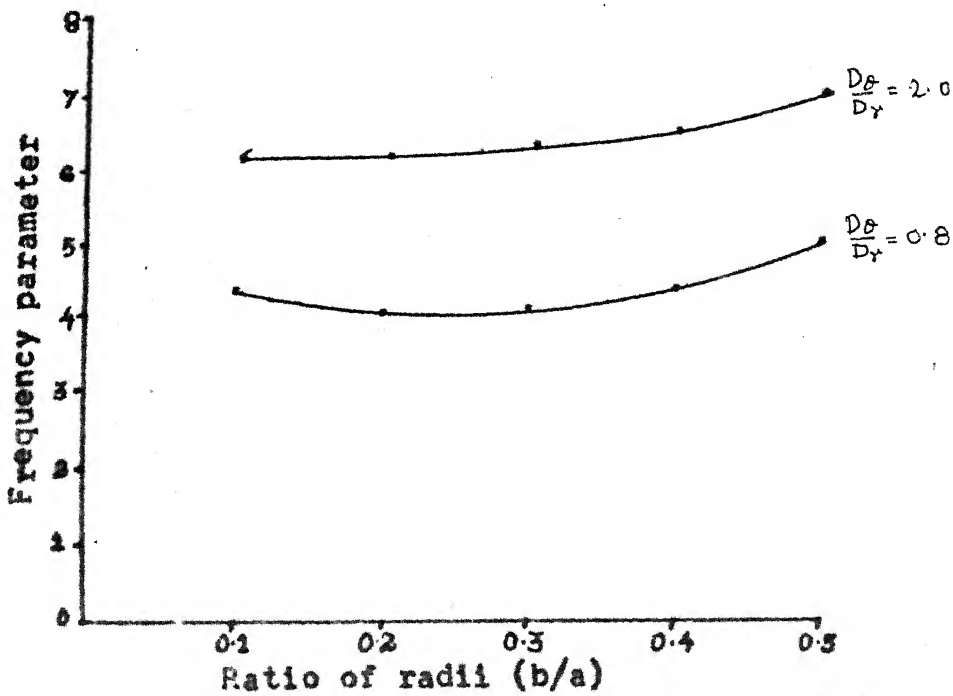

Fig. 4.11 Nondimensional frequency parameter Vs ratio of radii for S-F polar orthotropic plate

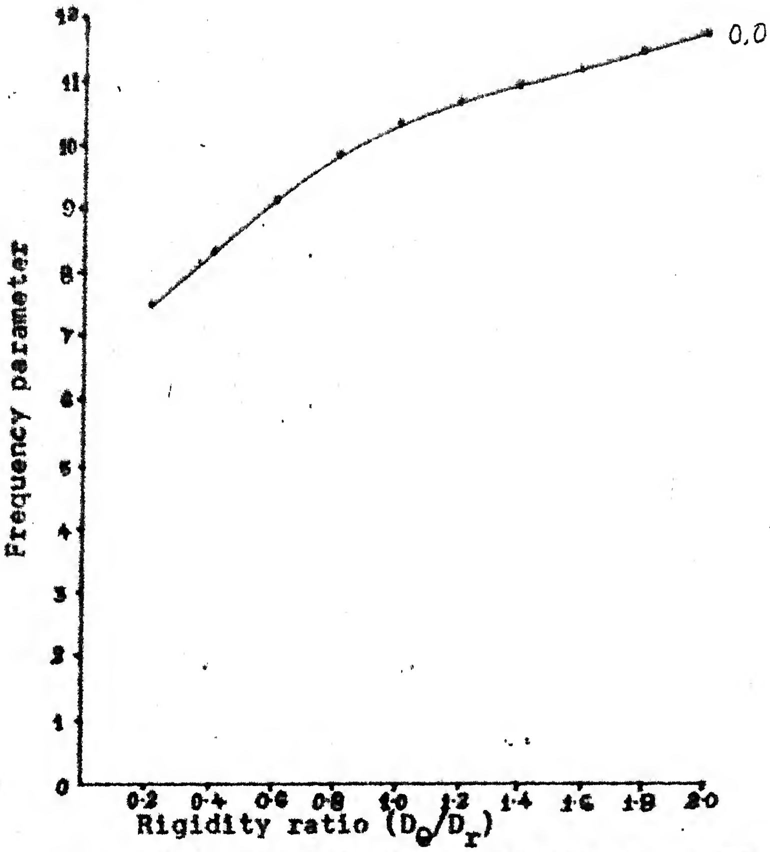

Fig. 4.12 Nondimensional frequency parameter Vs rigidity ratio for C-F polar orthotropic plate; $b/a = 0.1$

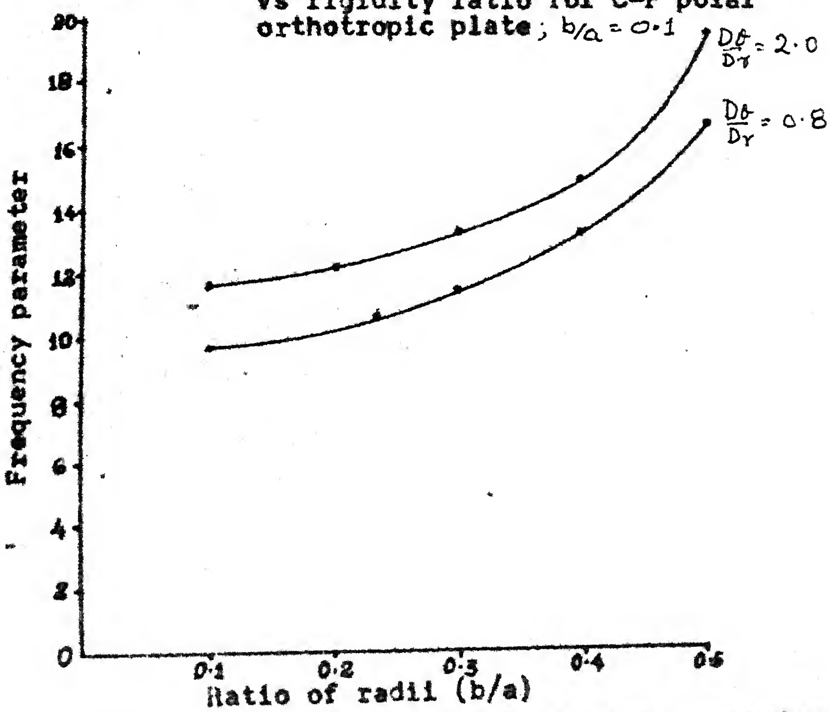

Fig. 4.13 Nondimensional frequency parameter Vs ratio of radii for C-F polar orthotropic plate

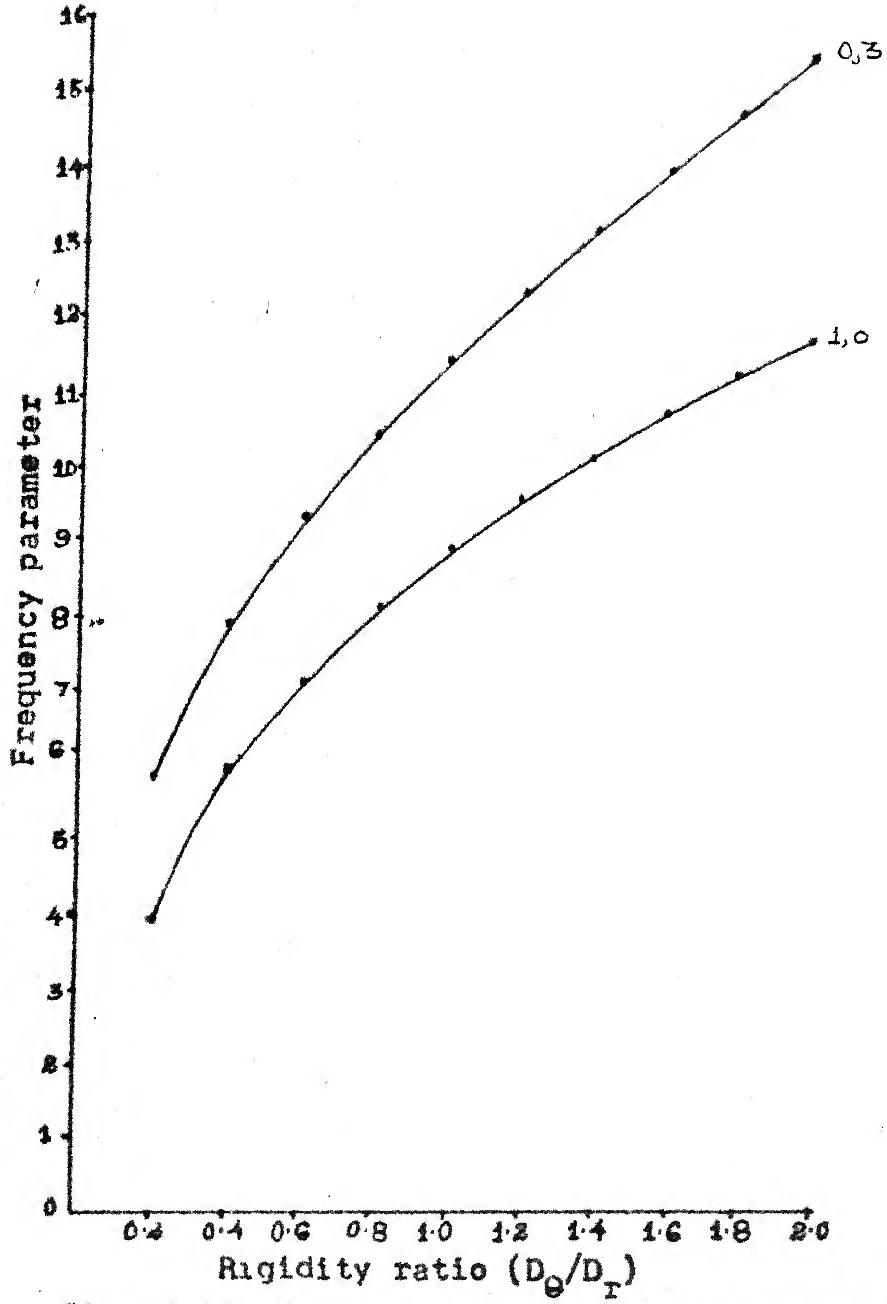

Fig. 4.14 Nondimensional frequency parameter Vs rigidity ratio for F-F polar orthotropic plate; $b/a = 0{\cdot}1$

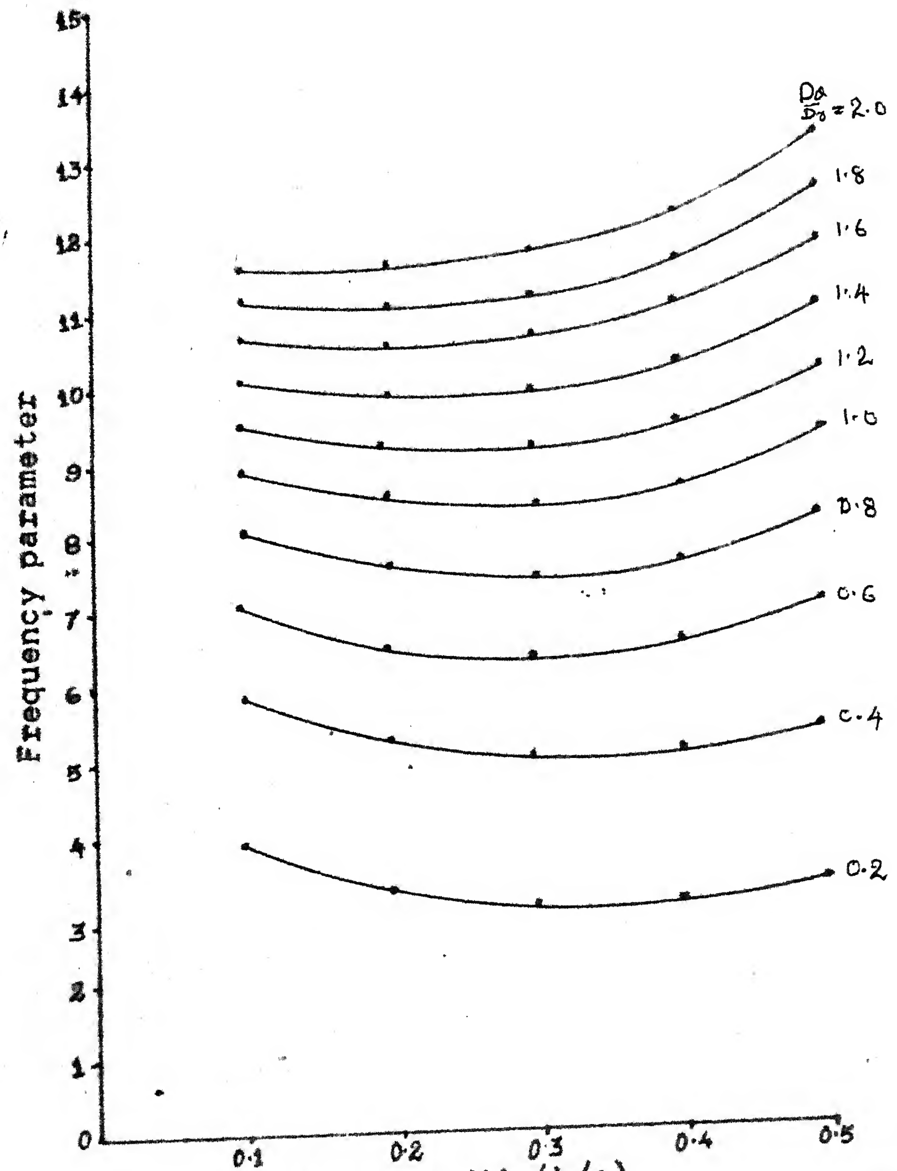

Fig. 4.15 Nondimensional frequency parameter Vs ratio of radii for F-F polar orthotropic plate; $n=1$, $m=0$.

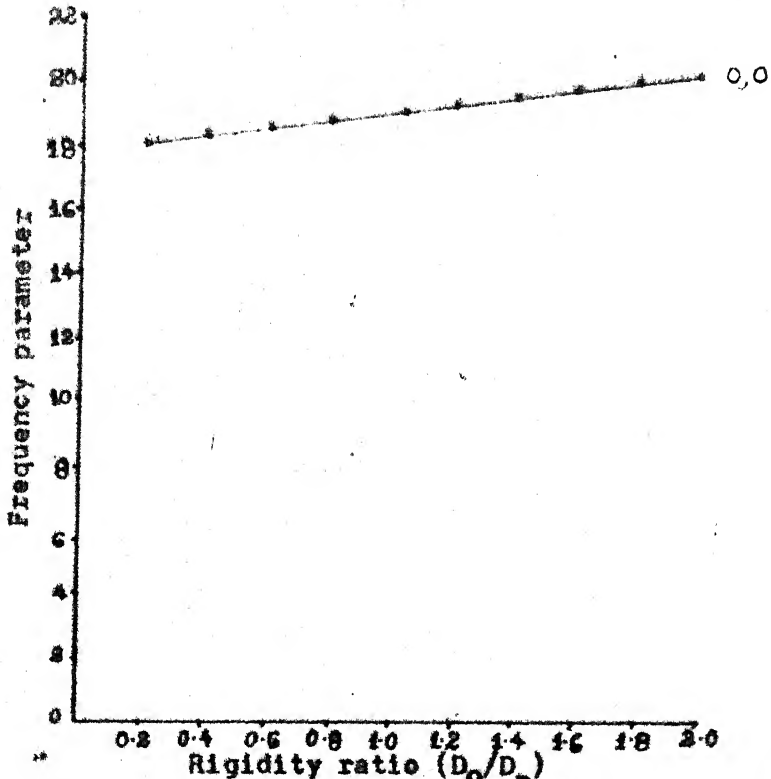

Fig. 4.16 Nondimensional frequency parameter Vs rigidity ratio for S–C polar orthotropic plate; b/a = 0·1

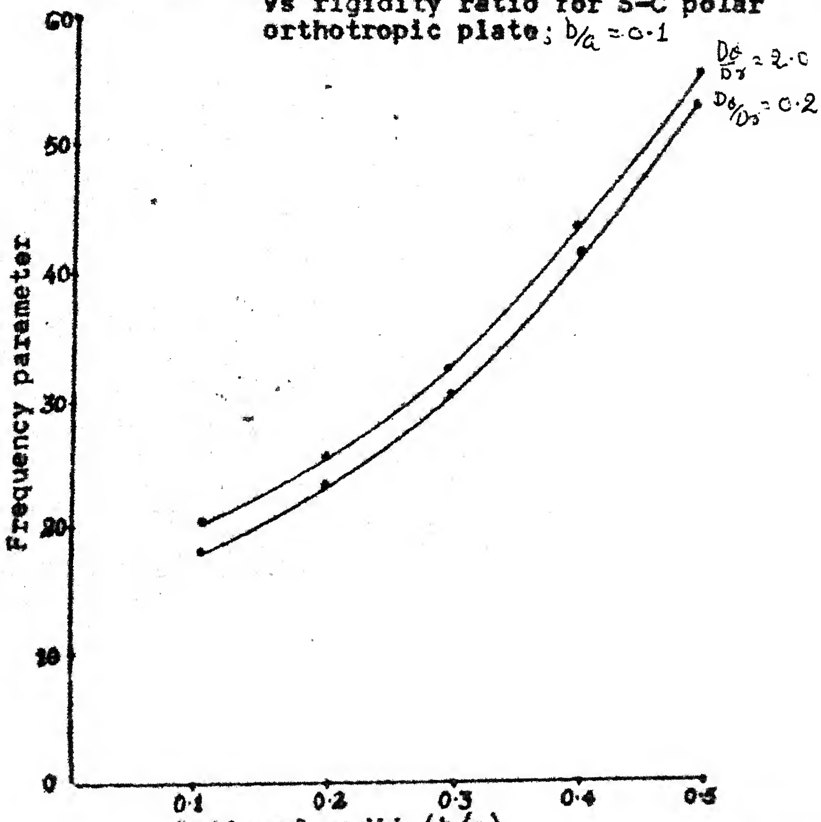

Fig. 4.17 Nondimensional frequency parameter Vs ratio of radii for S–C polar orthotropic plate

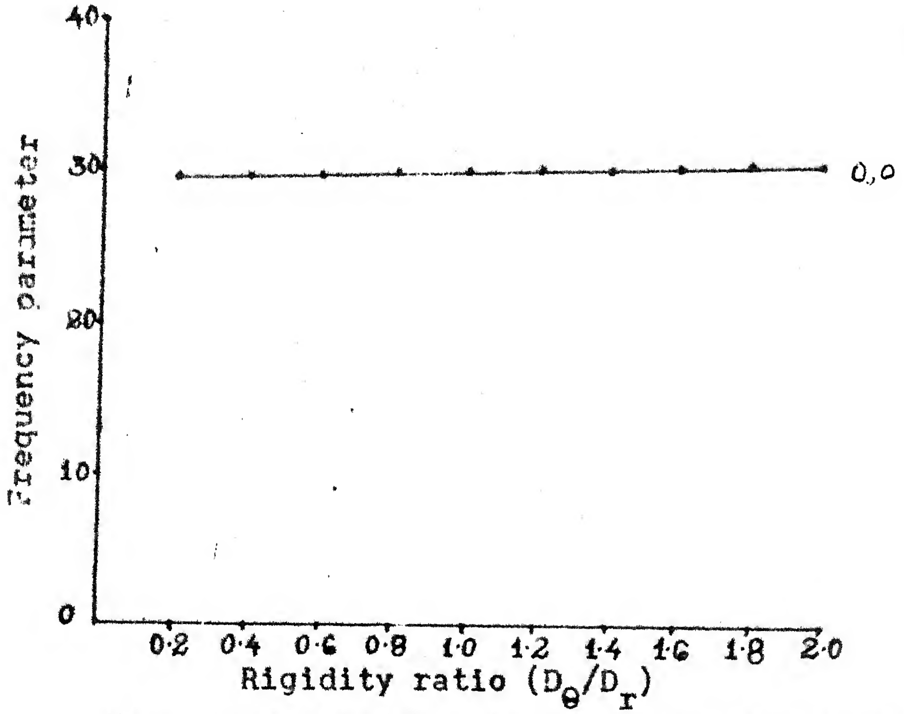

Fig. 4.18 Nondimensional frequency parameter Vs ridigity ratio for C-C polar orthotropic plate; $b/a = 0.1$

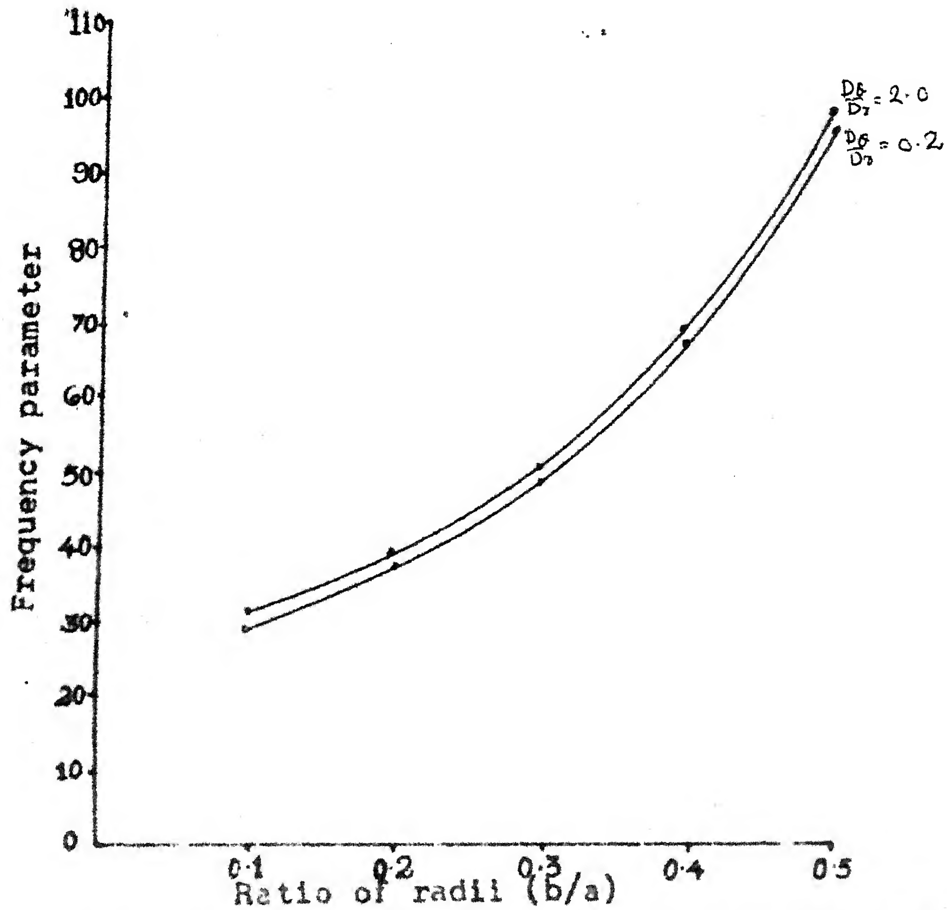

Fig. 4.19 Nondimensional frequency parameter Vs ratio of radii for C-C polar orthotropic plate

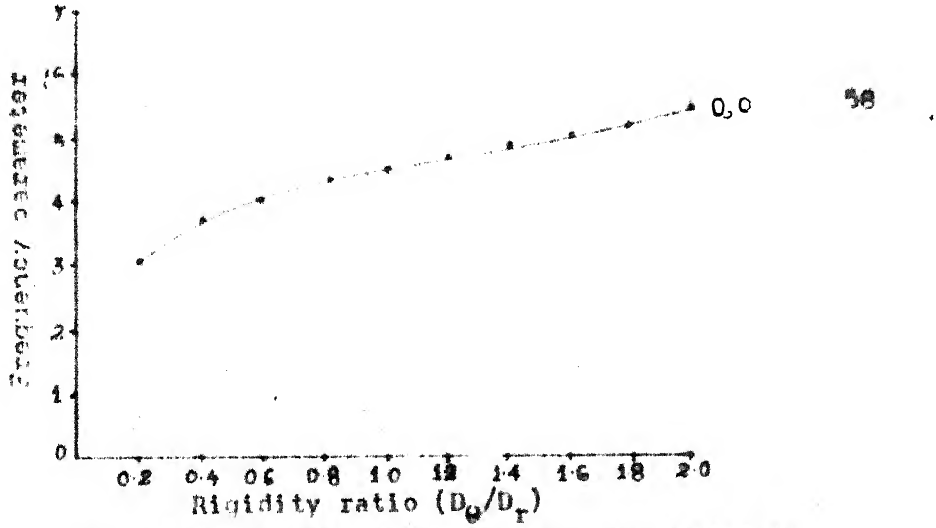

Fig. 4.20 Nondimensional frequency parameter Vs rigidity ratio for F–C polar orthotropic plate; $b/a = 0.1$

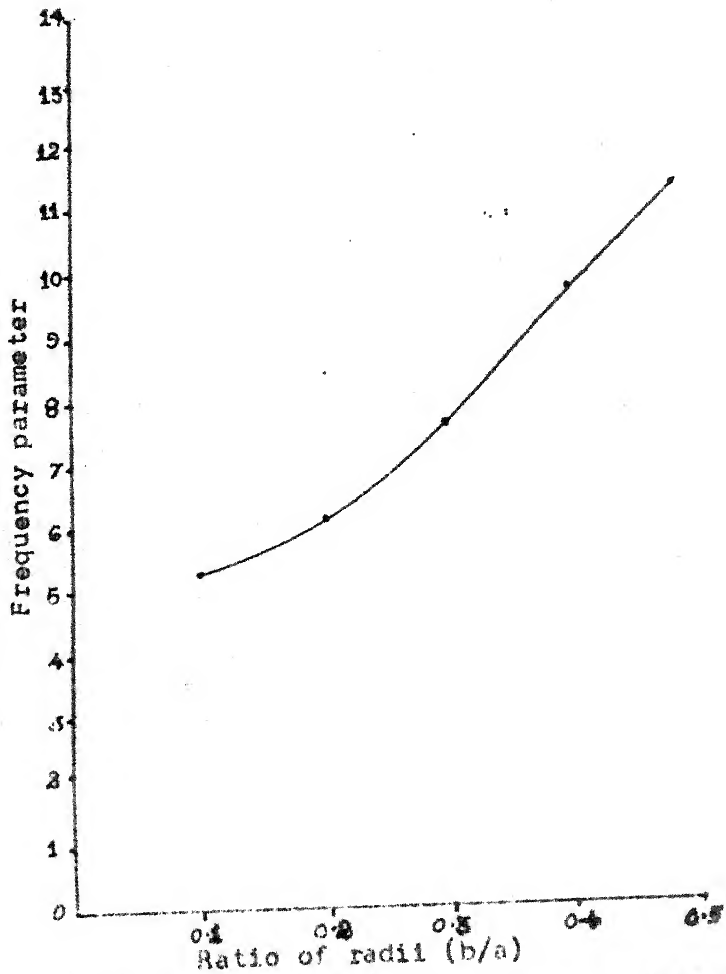

Fig. 4.21 Nondimensional frequency parameter Vs ratio of radii for F–C polar orthotropic plate; $D_\theta/D_r = 2.0$

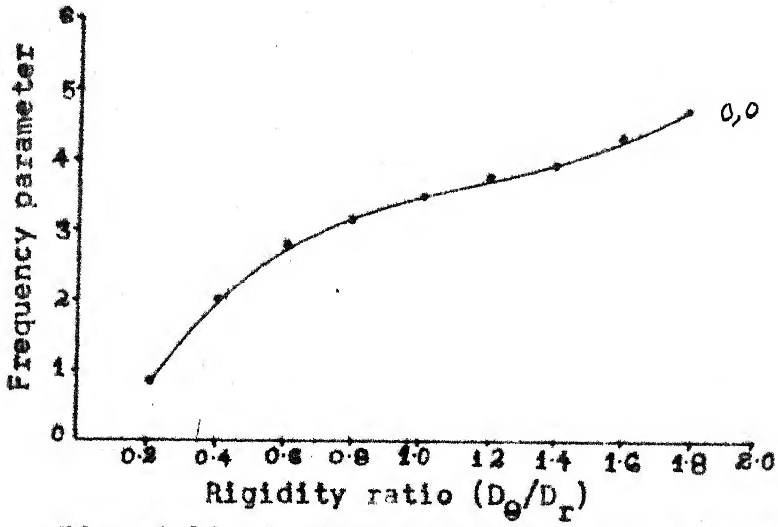

Fig. 4.24 Nondimensional frequency parameter Vs rigidity ratio for F-S polar orthotropic plate ; $b/a = 0.1$

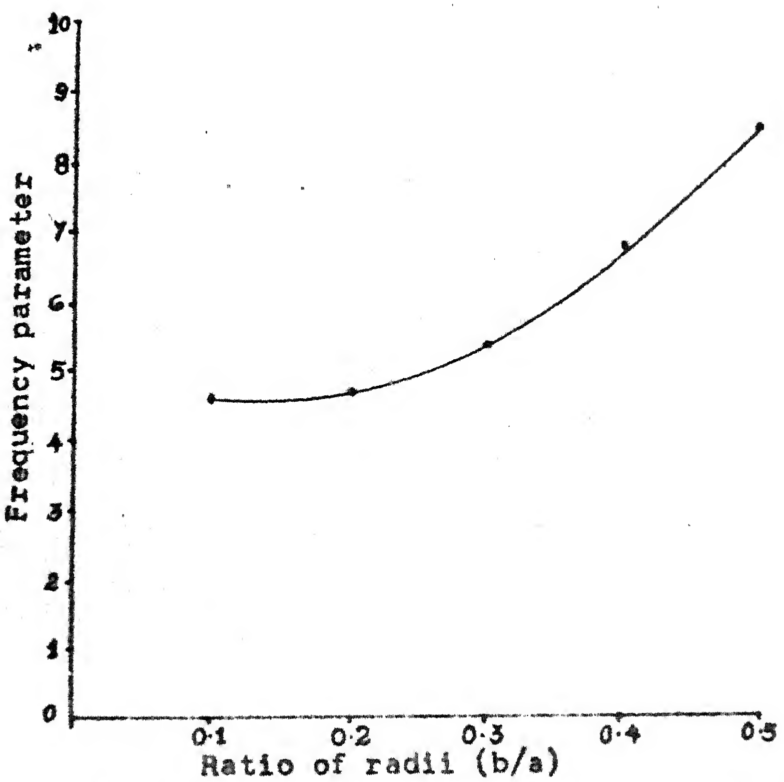

Fig. 4.25 Nondimensional frequency parameter Vs ratio of radii for F-S polar orthotropic plate ; $\frac{D_\theta}{D_r} = 2.0$

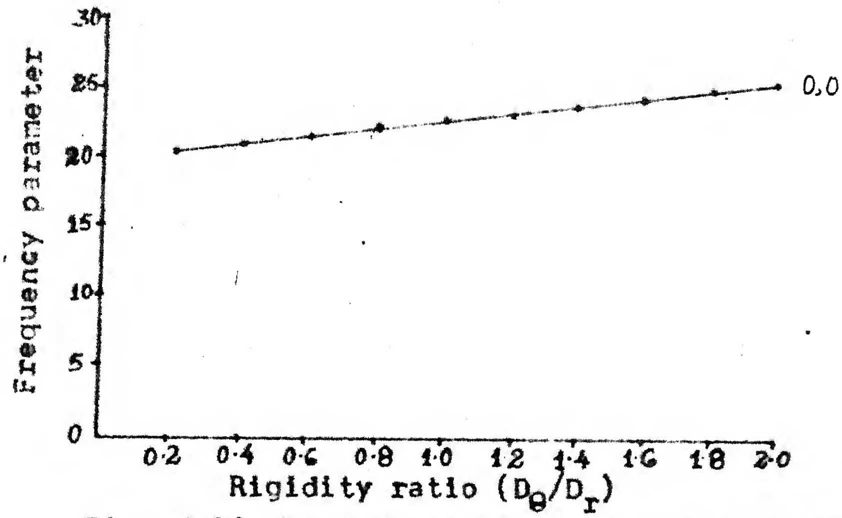

Fig. 4.26 Nondimensional frequency parameter Vs rigidity ratio for C-S polar orthotropic plate; $b/a = 0.1$

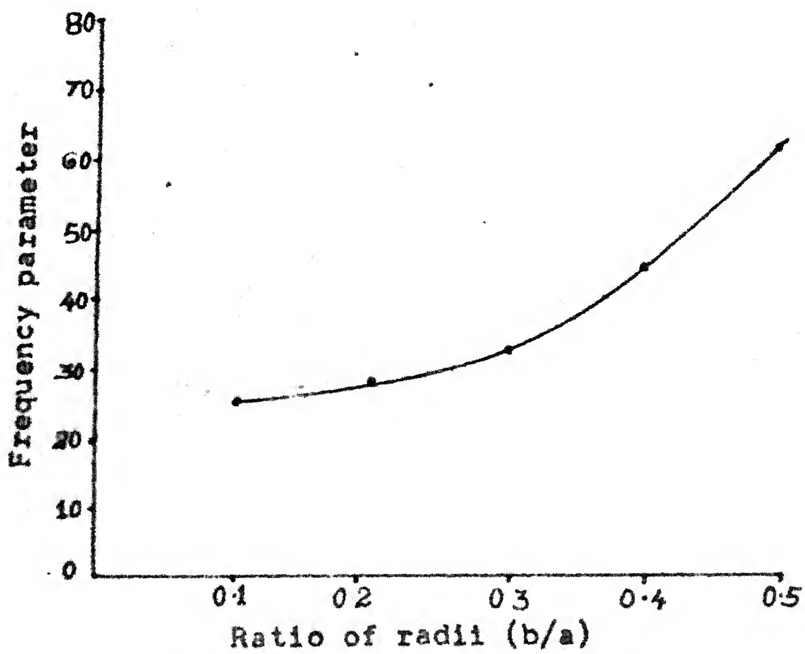

Fig. 4.27 Nondimensional frequency parameter Vs ratio of radii for C-S polar orthotropic plate; $D_\theta/D_r = 2.0$

frequencies are calculated for rigidity ratio ranging from 0.2 to 2.0 in steps of 0.2. Since numerical values are not available for comparison, the accuracy is checked by comparing the value obtained for rigidity ratio 1 (which corresponds to isotropic plate) with exact value of isotropic plate. The fundamental frequency parameter is given in Table 4.11 and the variation of fundamental frequency parameter with rigidity ratio is shown in Fig. 4.28. As expected the natural frequency increases with rigidity ratio and the variation is somewhat linear also. Table 4.12 gives the four lowest frequency parameters of orthotropic circular plate for a given rigidity ratio 0.8. Here also no numerical values are available for comparison, but it can be seen that the values obtained are less than and close to the corresponding frequencies of isotropic plate as expected because the rigidity ratio is 0.8.

Table 4.11

Variation of fundamental frequency parameter $[\omega a^2(\rho h/D_r)^{0.5}]$ with rigidity ratio (D_θ/D_r). Orthotropic circular plate; $D_{r\theta} = 0.35$, $\nu_\theta = 0.3$

Outer edge condition of the plate	Rigidity Ratio (D_θ/D_r)									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
Free	3.52	3.76	4.36	4.86	5.29 (5.24)	5.70	6.04	6.35	6.62	6.85
Clamped	8.63	9.30	9.75	10.11	10.42 (10.24)	10.74	11.03	11.29	11.56	11.78
Simply supported	3.34	3.86	4.30	4.63	4.97 (4.97)	5.25	5.45	5.74	5.97	6.17

Values in parentheses are exact values of frequency parameter of isotropic circular plate

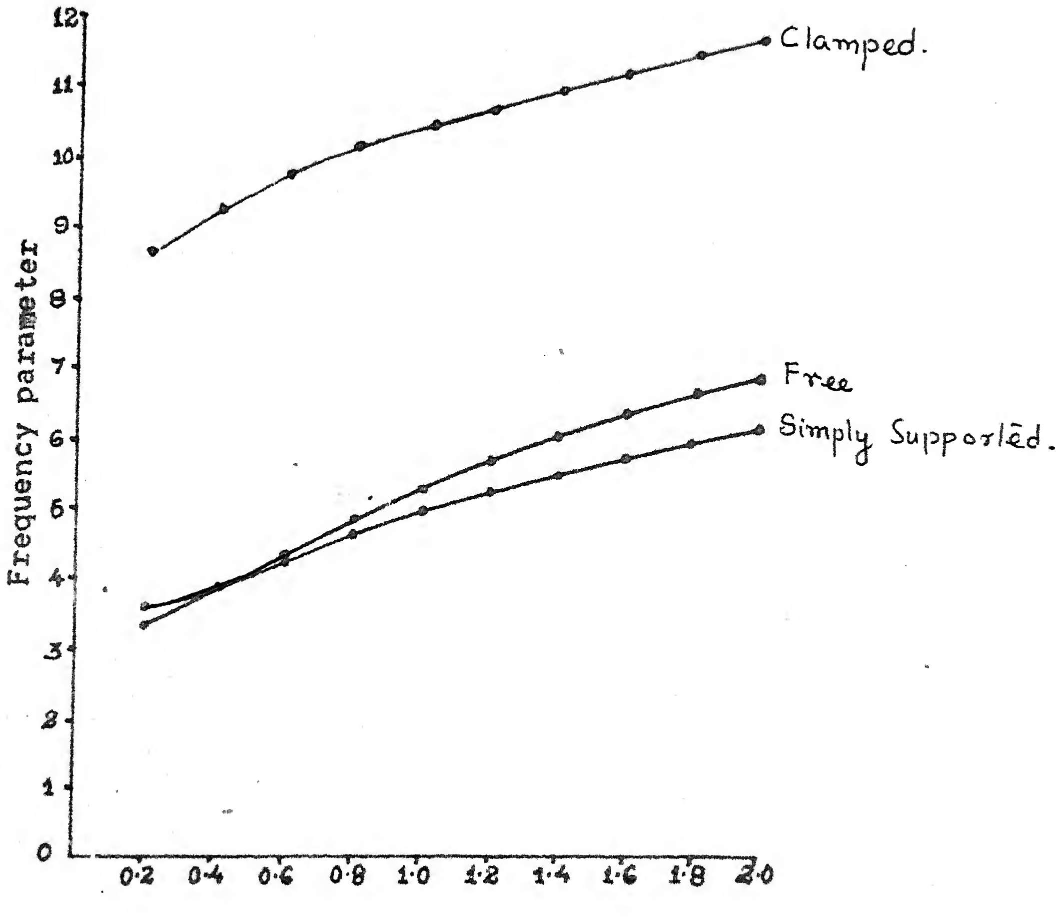

Fig. 4.28 Nondimensional frequency parameter Vs rigidity ratio for clamped,free and simply supported plates

Table 4.12

Frequency parameter $[\omega a^2(\rho h/D_r)]^{0.5}$ for orthotropic circular plates. $(D_\theta/D_r) = 0.8$, $D_{r\theta} = 0.35$ and $\nu_\theta = 0.3$

Outer edge condition of the plate	n	m	Frequency parameter
Free	0	2	4.86
	1	0	8.44
	1	1	19.45
	1	2	32.90
Clamped	0	0	10.11
	0	1	18.99
	0	2	23.33
	1	1	58.24
Simply supported	0	0	4.63
	0	1	13.48
	0	2	20.95
	1	1	44.06

CHAPTER 5

CONCLUSIONS

5.1 CONCLUSIONS

Vibration response of isotropic as well as polar orthotropic plates (circular and annular) are studied using finite element method. Annular finite elements having only four degrees of freedom, these being the displacements and slopes at inner and outer radii, are used, for the analysis. In Chapter 2, the vibration response of isotropic plates are studied and the estimates are compared with exact values. Then its applicability is extended to polar orthotropic plate also. In the case of isotropic plate, when the hole size is reduced to 0.001 of outside radius, the results obtained are seen to be very close to the exact values of solid circular plates. Hence, the vibration response of polar orthotropic solid circular plate is studied using the annular element itself. The material singularity at the centre of the circular plate, which is a major problem in the analysis by other methods, does not arise in the present analysis since here the solid plate is replaced by an annular plate having ratio of radii 0.001.

In the case of orthotropic plates, it is seen that the frequencies depend on rigidity ratio as well as hole size. The frequency increases with the increase in rigidity ratio, but

the effect of hole size is different for different edge conditions. The efficiency of using annular element is checked by comparing the results with the results obtained by using a sector element [7]. It is seen that the annular element with 4 degrees of freedom gives more accurate results then the results obtained by using sector elements having even 55 degrees of freedom. The results would have been better if the effect of rotary inertia and transverse shear deformations were included in the analysis.

In fairness it must be pointed out that the use of annular element described here is restricted to complete annular and circular plate, unlike the sector element [7], which can be used even for segments of annular and sector of circular plates.

5.2 SUGGESTIONS FOR FUTURE WORK

(i) In the present analysis the effect of rotary inertia and effect of additional deflection caused by shear force were not considered, it is worth to study the problem including these effects.

(ii) The applicability of this annular finite element can be extended to study vibration response of rotating polar orthotropic plates with any thickness variation in radial direction.

(iii) The effect of damping which is neglected in the present investigation can be included in the free and forced vibration response analysis of polar orthotropic plates.

(iv) An experimental study can be done to verify the results obtained by using the annular finite element.

REFERENCES

1. Thein Wah, 'Vibration of circular plates', The Journal of the acoustical Society of America, vol. 34, No.3, March, 1962, p. 275.

2. Timoshenko, S., Young, D.H., and Weaver, W., 'Vibration problems in engineering', John Wiley and Sons 1974, Fourth Edition.

3. Vogel, S.M., and Skinner, D.W., 'Natural frequencies of transversely vibrating uniform annular plates', Journal of Applied Mechanics, December, 1965, p. 926.

4. Zienkiewicz, O.C., Cheung, Y.K., 'The finite element method in structural and continuous mechanics', McGraw-Hill (1967).

5. Desai, C.S., and Abel, J.F., 'Introduction to the finite element method', Litton educational Publishing, Inc., 1972.

6. Ergatoudis , I., 'Curved, isoparametric, 'Quadrilateral' elements for finite element analysis', Int.J. Solids Structures, vol.4, 1968, p. 31.

7. Olson, M.D., Lindberg, G.M., 'Annular and circular sector finite elements for plate bending', Int.J. Mech.Sci, vol.12, 1970, p. 17.

8. Kirkhope, J., and Wilson, G.I., 'Vibration and stress analysis of thin rotating discs using annular finite elements', Journal of Sound and Vibration, vol.44, 1976, p. 461.

9. Kirmser, P.G., Huang, C.L., and Woo, H.K., 'Vibration of cylindrically orthotropic circular plates', AIAA Journal, 10,12 (TN) Dec. 1972, p. 1960.

10. Pandalai, K.A.V., and Sharad, A. Patel, 'Natural Frequencies of Orthotropic circular plates', AIAA J. (TN), vol.3, 1965, p. 780.

11. Joung, K.S., 'Derivation of Axisymmetric Flexural vibration equations of a cylindrically aeolotropic circular plates', AIAA J., vol.7, No.7, Part II, July 1969, p. 1386.

12. Minkarah, I.A., and Hoppmann, W.H., 'Flexural vibrations of cylindrically aeolotropic circular plates', J. of Acoustical Society of America, II 36: 1964, p.430.

13. Bellini, P.X. and Dzialo, F.J., 'Forced vibration of cylindrically orthotropic circular plates', J. of Acoustical Society of America, Part I, vol. 47, 1970, p. 940.

14. Huang, Chi Hung, 'Nonlinear axisymmetric flexural vibration equation of a cylindrically anisotropic circular plates', AIAA J., TN, vol. 10, 1972, p. 1378.

15. Nowinski, J.L., and Woodall, S.R., 'Finite vibrations of a free rotating anisotropic membrane', J. of Acoustical Society of America, vol. 36, No.11, Nov. 1964, p. 2113.

16. Vijayakumar, K., and Ramaiah, G.K., 'On the use of a coordinate transformation for analysis of axisymmetric vibration of polar orthotropic annular plates', Journal of Sound and Vibration, vol. 24, 1972, p. 165.

17. Ramaiah, G.K., and Vijayakumar, K., 'Natural frequencies of polar orthotropic annular plates', Journal of Sound and Vibration, vol.26, 1973, p. 517.

18. Ramaiah, G.K., and Vijayakumar, K., 'Estimation of higher natural frequencies of polar orthotropic annular plates', Journal of Sound and Vibration, vol. 32, 1974, p. 265.

19. Carrier, G.F., Ithaca, N.Y., 'The bending of the cylindrically aeolotropic plate', Journal of Applied Mechanics, September, 1944, A-129.

20. Sinha, P.K., 'Material characterisation and stress analysis of cylindrically reinforced composites', Ph.D. Thesis, Dept. of Mech. Engg., IIT Kanpur, India, 1979.

21. Narayana Raju, P., 'Vibrations of annular plates', Journal of the Aeronautical Society of India, vol.14, 1962, p. 37.

22. Guruswamy, P., and Yang, T.Y., 'A sector finite element for dynamic analysis of thick plates', Journal of Sound and Vibration, vol. 65, 1979, p. 505.